UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 57

ADVANCED COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In the 3+1 formalism, we represent spacetime using the line element

$$ds^2 = -N^2dt^2 + {}^{(3)}g_{ij}(dx^i - N^idt)(dx^j - N^jdt)\,,$$

where ${}^{(3)}g_{ij}(x^i)$ is the three metric on constant time t hypersurfaces Σ, the lapse function $N(t,x^i)$ defines the change in the proper time and the shift vector $N^i(t,x^i)$ gives the change in the spatial coordinates for a 'normal' trajectory defined along $n_\mu = (-N,\ 0,\ 0,\ 0)$. The Einstein equations in this metric become [*do not attempt to derive these*]:

$$\begin{gathered} {}^{(3)}R + \tfrac{2}{3}K^2 - \tilde{K}_{ij}\tilde{K}^{ij} = 16\pi G\rho\,, \qquad \tilde{K^j}_{i|j} - \tfrac{2}{3}K_{|i} = 8\pi G\mathcal{J}_i\,. \\ \dot{K} + N^iK_{,i} + N^{|i}{}_{|i} - N({}^{(3)}R + K^2) = 8\pi GN\left(\tfrac{1}{2}S - \tfrac{3}{2}\rho\right)\,, \\ \dot{\tilde{K}}{}^i{}_j + N^k\tilde{K}^i_{j|k} - N^i{}_{|k}\tilde{K}^k_j + N^k{}_{|j}\tilde{K}^i_k + N^{|i}{}_{|j} - \tfrac{1}{3}N^{|k}{}_{|k}\delta^i{}_j - N({}^3\tilde{R}^i_j + K\tilde{K}^i_j) = -8\pi GN\tilde{S}^i_j\,, \end{gathered}$$

where $|$ denotes the covariant derivative in Σ, the intrinsic curvature is ${}^{(3)}R_{ij}$ (with Ricci scalar ${}^{(3)}R$), ρ, $\mathcal{J}_i$ and S_{ij} are defined below, and the extrinsic curvature K_{ij} splits into trace and traceless parts, respectively,

$$K \equiv {}^{(3)}g_{ij}K^{ij}\,, \qquad \tilde{K}_{ij} \equiv K_{ij} - \tfrac{1}{3}{}^{(3)}g_{ij}K\,.$$

(i) For a scalar field ϕ with Lagrangian $\mathcal{L}_{\rm m} \equiv \sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$, with conjugate momentum $\Pi \equiv \partial\mathcal{L}_{\rm m}/\partial\dot{\phi} = n^\mu\partial_\mu\phi = (\dot{\phi} + N^i\partial_i\phi)/N$, derive the energy-momentum tensor $T^{\mu\nu}$ and find its 3+1 projected components: the energy density,

$$\rho = n_\mu n_\nu T^{\mu\nu} = N^2T^{00} = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}\partial_i\phi\,\partial^i\phi + V(\phi)\,,$$

the momentum density $\mathcal{J}_i = -\Pi\partial_i\phi$, and the stress tensor S_{ij} which has the trace and traceless parts, respectively,

$$S \equiv {}^{(3)}g^{ij}S_{ij} = \tfrac{3}{2}\Pi^2 - \tfrac{1}{2}\partial_k\phi\,\partial^k\phi - 3V(\phi)\,, \qquad \tilde{S}_{ij} = \tfrac{1}{2}(\partial_i\phi\,\partial_j\phi - \tfrac{1}{3}\partial_k\phi\,\partial^k\phi\;\delta_{ij})\,.$$

(ii) The extrinsic curvature is defined by

$$K_{ij} \equiv -n_{(i;j)} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)\,.$$

Consider the conformal 3-metric ${}^{(3)}\tilde{g}_{ij} = a^{-2}(t,\,x^i){}^{(3)}g_{ij}$ where $a^6 \equiv {}^{(3)}g = \det({}^{(3)}g_{ij})$ and, hence or otherwise, take the trace of the extrinsic curvature expression to find

$$K \equiv {}^{(3)}g^{ij}K_{ij} = -\frac{1}{2N}\left(\frac{{}^{(3)}\dot{g}}{{}^{(3)}g} + 2N^i_{|i}\right)\,.$$

In the context of an expanding universe (setting $N^i = 0$), argue that $-K/3$ can be interpreted as a locally defined Hubble parameter $H(t,x^i)$. [*Hint: You may use that* $\mathrm{Tr}(A^{-1}dA/dt) = d(\ln(\det A))/dt$ *for any matrix A with non-vanishing determinant.*]

(iii) In the long wavelength approximation we neglect second order spatial gradients. Rewrite the Einstein equations in long wavelength form with the scalar field from (i), using

$K = -3H$ and taking zero shift $N^i = 0$. Hence, show that the traceless part of the extrinsic curvature has the general solution $\tilde{K}^i_j \approx C^i_j(x)\, a^{-3}$.

(iv) Use the long wavelength Einstein equations from (iii) to establish that the nonlinear inhomogeneous variable

$$\zeta_i \equiv -\frac{\partial_i a}{a} + \frac{H}{\Pi}\partial_i \phi\,,$$

is conserved on superhorizon scales, that is, $\dot{\zeta}_i = 0$ for $k \ll aH$. [*Hint: Neglect all* $\tilde{K}^i_j$ *terms and differentiate* $H^2 = \frac{8\pi G}{3}(\frac{1}{2}\Pi^2 + V(\phi))$ *to determine* $\dot{\Pi}$ *and* $\partial_i \Pi$.]

Consider the linear adiabatic perturbation $\zeta \equiv \Psi - \frac{1}{3}\delta\rho/(\bar{\rho} + \bar{P})$, which we have defined at linear order using $a = \bar{a}(1 - \Psi)$ expanding the field ϕ as $\phi = \bar{\phi} + \delta\phi$ with homogeneous (background) scale factor $\bar{a}$, energy density $\bar{\rho}$ and pressure $\bar{P}$. Show that $\zeta_i \approx \partial_i \zeta$ to linear order and briefly discuss the implications for single field inflation.

2

In synchronous gauge for linear perturbations about a flat ($k = 0$) FRW background, the metric is taken to be

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] ,$$

where $|\det(h_{ij})| \ll 1$ and the relevant connections are $\Gamma^0_{00} = a'/a$, $\Gamma^0_{0i} = \Gamma^i_{00} = 0$,

$$\Gamma^0_{ij} = \frac{a'}{a}\left[\delta_{ij} + h_{ij}\right] + \frac{1}{2}h'_{ij}\,, \qquad \Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}\,, \qquad \Gamma^i_{jk} = \frac{1}{2}[h_{ij,k} + h_{ik,j} - h_{jk,i}]\,,$$

with primes (e.g. a') denoting differentiation with respect to the conformal time τ. We define the scalar trace $h = h_{ii}$ and anisotropic scalar h_s.

(i) Assume the universe is filled with a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}\,,$$

where u^μ is the fluid 4-velocity, ρ is the energy density and P is the pressure, with the latter related through an equation of state $P = w\rho$. In a comoving synchronous frame (i.e. $u^\mu = a^{-1}(1,\ \mathbf{v})$ with $|\mathbf{v}| \ll 1$), *linearise* the energy-momentum tensor to find that in Fourier space we have

$$T^{00} = \frac{1}{a^2}\bar\rho(1+\delta)\,, \qquad T^{0i} = \frac{1}{a^2}(1+w)\bar\rho\, ik_i\theta\,, \qquad T^{ij} = \frac{1}{a^2}w\bar\rho\left[(1+\delta)\delta_{ij} - h_{ij}\right]\,,$$

with k_i the components of the comoving wavevector $\mathbf{k}$ ($k = |\mathbf{k}|$), the background density $\bar\rho = \bar\rho(t)$, and where suitable definitions should be given for the density perturbation δ and the velocity potential θ. Show that the energy conservation equation $T^{0\mu}{}_{;\mu} = 0$ yields

$$\delta' - (1+w)k^2\theta + \tfrac{1}{2}(1+w)h' = 0\,.$$

You do *not* need to derive the corresponding momentum conservation equation, which is given as $\theta' + (1-3w)\frac{a'}{a}\theta + \frac{w}{1+w}\delta = 0$.

(ii) When solving the Boltzmann equation near decoupling for the photon-baryon system, we find the moment expansion yields the following equations for the photon density δ_γ, velocity θ_γ and shear viscosity σ_γ and for the baryon density $\delta_{\rm b}$ and velocity $\theta_{\rm b}$ [*do not derive these equations*]:

$$\begin{aligned}
&\delta'_\gamma - \tfrac{4}{3}k^2\theta_\gamma = -\tfrac{2}{3}h'\,, \qquad \theta'_\gamma + \tfrac{1}{4}\delta_\gamma - \sigma_\gamma = -an_{\rm e}\sigma_{\rm T}(\theta_\gamma - \theta_{\rm b})/k^2\,,\\
&\sigma'_\gamma + \tfrac{4}{15}k^2\theta_\gamma = -\tfrac{4}{15}h'_{\rm S} - an_{\rm e}\sigma_{\rm T}\sigma_\gamma\,,\\
&\delta'_{\rm b} + k^2\theta_{\rm b} = -\tfrac{1}{2}h'\,, \qquad \theta'_{\rm b} + \tfrac{a'}{a}\theta_{\rm b} + c_{\rm s}^2\delta_{\rm b} = -Ran_{\rm e}\sigma_{\rm T}(\theta_{\rm b} - \theta_\gamma)/k^2\,,
\end{aligned}$$

where we have terminated the series at the third moment $\ell = 3$ and the ratio R is given in terms of the relative background photon and baryon densities $R = (4/3)\bar\rho_\gamma/\bar\rho_{\rm b}$.

Discuss the comparison between these equations and those obtained assuming the baryons and photons are decoupled perfect fluids as in part (i). In particular, explain the origin and nature of the collision terms. Provide a brief argument why we can ignore higher moments ($\ell \geqslant 2$) for non-relativistic particles.

(iii) For initially adiabatic perturbations, while the photons and baryons are tightly coupled we will have $\delta_\gamma \approx \frac{4}{3}\delta_{\rm b}$ and $\theta_\gamma \approx \theta_{\rm b}$. Show that in this limit, the photon and baryon evolution equations can be combined to become

$$\delta_\gamma' = \tfrac{4}{3}k^2\theta_\gamma - \tfrac{2}{3}h' \,, \qquad \theta_\gamma' = -3\tilde{c}_{\rm s}^2\left(\tfrac{1}{4}\delta_\gamma - \sigma_\gamma\right) - \tfrac{3\tilde{c}_{\rm s}^2}{R}\left(\tfrac{a'}{a}\theta_\gamma + \tfrac{3}{4}c_{\rm s}^2\delta_\gamma\right)\,,$$

where the effective combined sound speed is given by $\tilde{c}_{\rm s}^2 \equiv \frac{1}{3}\frac{R}{1+R} = \frac{1}{3}\left(1+\frac{3}{4}\frac{\bar\rho_{\rm b}}{\bar\rho_\gamma}\right)^{-1}$.

(iv) In the limit that decoupling is in the matter era (cold dark matter $\Omega_{\rm c} \approx 1$ and $a \propto \tau^2$) with $\bar\rho_\gamma \gg \bar\rho_{\rm b}$, show that the tight coupling equations can be combined to yield

$$\delta_\gamma'' - \tfrac{4}{3}k^2\sigma_\gamma + \tfrac{1}{3}k^2\delta_\gamma = -\tfrac{2}{3}h''\,.$$

Ignoring the slowly-varying inhomogenous terms (involving h and $h_{\rm S}$) and assuming that $\sigma_\gamma' \approx 0$, show that this becomes

$$\delta_\gamma'' + \frac{4}{15}\tau_{\rm c}k^2\delta_\gamma' + \frac{1}{3}k^2\delta_\gamma = 0\,.$$

where $\tau_{\rm c} = (a\sigma_{\rm T}n_{\rm e})^{-1}$. Find general solutions of this equation (ignoring terms of $\mathcal{O}(\tau_{\rm c}^2)$) and show that for subhorizon scales they take the following approximate form

$$\delta_\gamma(\mathbf{k},\tau) \approx \left[A(\mathbf{k})\cos(k\tau/\sqrt{3}) + B(\mathbf{k})\sin(k\tau/\sqrt{3})\right]\,\exp(-k^2/k_{\rm D}^2)\,,$$

where $k_{\rm D}$ is a time-dependent damping scale which you should specify. Briefly comment on the implications of this solution for the temperature anisotropy $\frac{\Delta T}{T}$ power spectrum.

3

(a) Consider the following Lagrange density up to 3rd order in perturbation theory

$$\mathcal{L} = \frac{1}{2}\dot{\zeta}^2 - \frac{1}{2}(\partial\zeta)^2 + \alpha\dot{\zeta}^3 + \beta\zeta(\partial\zeta)^2 + \gamma(\partial^{-2}\zeta)(\dot{\zeta})^2 \,.$$

Calculate the conjugate momentum π

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\zeta}} \,.$$

Hence, calculate the Hamiltonian density for this action $\mathcal{H}(\pi,\zeta)$ to *third* order in perturbation theory. Identify the *interaction Hamiltonian density* $\mathcal{H}_{int}$.

(b) A *rescaling* is defined to be $\mathbf{x} \to \lambda\mathbf{x}$ where $\lambda > 0$ is a real constant. State the condition for the 2-point correlation function in $\mathbf{x}$ space to be *scale-invariant*, and then show that this implies that the power spectrum must scale like

$$P(k) \propto \frac{1}{k^3}.$$

(c) The shape function of a single scalar field slow roll inflationary model is given by

$$F(k_1,k_2,k_3) = (2\pi)^3 \frac{H^4}{M_p^4}\frac{1}{4\epsilon^2}\frac{1}{k_1^3k_2^3k_3^3}\left[\frac{\eta}{8}\sum_i k_i^3 + \frac{\epsilon}{8}\left(-\sum_i k_i^3 + \sum_{i\neq j} k_i k_j^2 + \frac{8}{K}\sum_{i>j} k_i^2 k_j^2\right)\right],$$

where ϵ and η are the slow-roll parameters and $K = k_1 + k_2 + k_3$. Find the *squeezed* limit of this shape function. What is the geometrical meaning of this limit?

(d) Consider a field $\Phi(\mathbf{x})$, defined by the following ansatz

$$\Phi(\mathbf{x}) = \Phi_G(\mathbf{x}) + f_{NL}^{local}(\Phi_G(\mathbf{x})^2 - \langle\Phi_G(\mathbf{x})^2\rangle),$$

where f_{NL}^{local} is a constant and $\Phi_G(\mathbf{x})$ is a Gaussian random field in the sense that its Fourier transform coefficients $\Phi_G(\mathbf{k})$

$$\Phi_G(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\Phi_G(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

are drawn from a Gaussian probability distribution function, with its linear power spectrum given by

$$\langle\Phi_G(\mathbf{x})\Phi_G(\mathbf{x}')\rangle \equiv \int \frac{d^3k}{(2\pi)^3}P(k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}.$$

Show that the leading higher order (i.e. beyond the 2-pt correlation funcation) correlation function for $\Phi(\mathbf{x})$ is given by

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = f_{NL}^{local}\left[2(2\pi)^3P(k_1)P(k_2)\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) + \mathrm{sym}\right].$$

4

Consider the following term in the interaction Hamiltonian for a non-canonical theory of inflation

$$H_{int}(\tau) = \int d^3x \, \frac{a(\tau)\epsilon}{c_s^2}(\epsilon + 1 - c_s^2)\zeta(\mathbf{x},\tau)(\partial\zeta(\mathbf{x},\tau))^2 \,,$$

where τ is conformal time, and ∂ denotes the spatial derivative. The slow-roll parameter ϵ and the sound speed $c_s^2 \leqslant 1$ are varying very slowly with time, so for the purpose of this calculation you can assume that they are constants.

During inflation, we can expand the *interaction picture* field ζ_I in the following way

$$\zeta_I(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\left[a_I(\mathbf{k})u_k^*(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + a_I^\dagger(\mathbf{k})u_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] = \zeta_I^+(\mathbf{x},\tau) + \zeta_I^-(\mathbf{x},\tau)\,,$$

where the mode function has the following solution

$$u_k(\tau) = \frac{H}{\sqrt{4\epsilon c_s k^3}}(1 - ikc_s\tau)e^{ic_sk\tau}.$$

(i) Using this interaction Hamiltonian, show that the 3-point correlation function at $\tau \to 0$

$$\langle\zeta(\mathbf{k}_1,\tau)\zeta(\mathbf{k}_2,\tau)\zeta(\mathbf{k}_3,\tau)\rangle$$

$$= \mathrm{Re}\left\langle\left[-2i\zeta_I(\mathbf{k}_1,\tau)\zeta_I(\mathbf{k}_2,\tau)\zeta_I(\mathbf{k}_3,\tau)\int_{-\infty(1+i\epsilon)}^{\tau} d\tau' \, a(\tau')H_{int}^I(\tau')\right]\right\rangle$$

is given by

$$\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle = \frac{H^4}{16\epsilon^2 c_s^4}(\epsilon + 1 - c_s^2)(2\pi)^3\frac{1}{k_1^3k_2^3k_3^3}\delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$$

$$\times\left((\mathbf{k}_1\cdot\mathbf{k}_2)\left(-K + \frac{k_1k_2 + k_1k_3 + k_2k_3}{K} + \frac{k_1k_2k_3}{K^2}\right) + 1 \to 3 + 2 \to 3\right),$$

where $K = k_1 + k_2 + k_3$. [You may assume that the scale factor $a(\tau) = -1/(H\tau)$ and τ runs from $-\infty < \tau < 0$.]

(ii) Write down the leading contribution to the 3-point correlation function for this interaction term in the following two limits, assuming that $\epsilon \approx 0.01$,

- $c_s^2 \to 1$,
- $c_s^2 \ll 1$.

What is the ratio of the amplitudes of the 3-point correlation function generated by the above two terms? Compare and comment on their relative magnitude as a function of c_s^2.

END OF PAPER