UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 9:00 am to 12:00 pm

PAPER 56

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain the following terms: *covariant derivative*, *connection components*.

(b) What does it mean for a connection to be *torsion-free*? Define the *Levi-Civita connection* and derive the formula

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$

(c) A 2d Riemannian manifold has metric

$$ds^2 = dr^2 + f(r)^2 \, d\phi^2$$

where ϕ is periodically identified with period 2π (so curves of constant r are circles).

(i) Determine the necessary and sufficient conditions on $f(r)$ for the circle $r = r_0$ to have the property that all vectors are invariant under parallel transport around the circle.

(ii) Deduce that there is a 2-parameter family of functions $f(r)$ for which all circles of constant r have this property.

(iii) For this 2-parameter family, show that the metric is locally isometric to the Euclidean metric. Is it globally isometric?

2

Let T^a be tangent to a 1-parameter family of timelike geodesics of the Levi-Civita connection, parameterized by proper time. Let S^a be a deviation vector for this family.

(a) Explain the term *deviation vector*. Explain why $[S, T] = 0$.

(b) State and prove the *geodesic deviation equation*. You may assume the definition of the Riemann curvature tensor:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

(c) Prove that if S^a and T^a are orthogonal at one point along a geodesic γ (belonging to the 1-parameter family) then they are orthogonal everywhere along γ.

(d) Suppose that spacetime is four-dimensional with Riemann tensor

$$R_{abcd} = \frac{1}{12} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right)$$

Show that R must be constant.

(e) Assume that S^a is orthogonal to T^a. Let $f = S_a S^a$. Show that, in the above spacetime, the following quantity is constant along a geodesic γ in the family

$$K = (\nabla_T S)_a (\nabla_T S)^a - \frac{1}{12} R f$$

(f) Obtain a second order differential equation for the evolution of f along γ. Hence deduce that if $R > 0$ then geodesics which are close initially will diverge exponentially. What happens if $R < 0$?

3

In the study of linearized perturbations of Minkowski spacetime, it is assumed that there exist global coordinates x^μ with respect to which the metric has components $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the components of $h_{\mu\nu}$ have absolute values much smaller than 1. Indices are raised with $\eta^{\mu\nu}$ and lowered with $\eta_{\mu\nu}$.

(a) Let $h = h^\rho{}_\rho$ and $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h\,\eta_{\mu\nu}$. By imposing the gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$, derive the linearized Einstein equation in the form

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu}$$

You may use the following formula, valid in a coordinate basis:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}$$

(b) Consider a vacuum plane gravitational wave solution with

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left(H_{\mu\nu} e^{ik_\rho x^\rho}\right)$$

where $H_{\mu\nu}$ is a constant complex matrix and k_ρ a constant covector. What restrictions must $H_{\mu\nu}$ and k_ρ obey in the above gauge?

Explain why there is a residual gauge freedom $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ provided ξ_μ satisfies a certain condition. Show that this condition is satisfied by

$$\xi_\mu = \mathrm{Re}\left(X_\mu e^{ik_\rho x^\rho}\right)$$

where X_μ is constant. Now show that X_μ can be chosen to bring $H_{\mu\nu}$ to the form

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Use your results to justify the statements that gravitational waves travel at the speed of light, are transverse, and have 2 independent polarizations.

(c) (i) Explain briefly why it is not possible to define an energy-momentum tensor for the gravitational field in General Relativity.

(ii) Explain how to define a symmetric tensor $t_{\mu\nu}$ that is quadratic in the linearized gravitational field $h_{\mu\nu}$ and conserved $\partial^\mu t_{\mu\nu} = 0$. (You are not expected to give detailed formulae for the curvature tensors expanded to second order.)

(iii) Why is $t_{\mu\nu}$ unsatisfactory as a definition of an energy-momentum tensor for $h_{\mu\nu}$?

4

(a) (i) What is a Killing vector field?

(ii) Show that if a spacetime admits a Killing vector field then along any geodesic there is a conserved quantity.

(iii) Write down 3 linearly independent Killing vector fields of the metric

$$ds^2 = A(z)^2 \left(-dt^2 + dx^2 + dy^2\right) + dz^2$$

where $A(z)$ is a positive function.

(b) An orthonormal basis for the above metric is defined by

$$e^0 = A\,dt \qquad e^1 = A\,dx \qquad e^2 = A\,dy \qquad e^3 = dz$$

Determine the connection 1-forms ${\omega^\mu}_\nu$ satisfying $de^\mu = -{\omega^\mu}_\nu \wedge e^\nu$ and the curvature 2-forms defined by

$${\Theta^\mu}_\nu = d{\omega^\mu}_\nu + {\omega^\mu}_\rho \wedge {\omega^\rho}_\nu$$

(c) The *null energy condition* states that the energy-momentum tensor should satisfy $T_{ab}k^a k^b \geqslant 0$ for any null vector k^a. Show that, if the above metric satisfies the Einstein equation with an energy-momentum tensor satisfying the null energy condition then

$$\frac{d^2}{dz^2} \log A(z) \leqslant 0$$

END OF PAPER