UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 9:00 am to 11:00 am

PAPER 55

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Find the structure constants of the three–dimensional Lie algebra $\mathfrak{g}$ generated by matrices

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie group G corresponding to $\mathfrak{g}$ is the multiplicative group of real matrices of the form

$$\mathsf{g} = \begin{pmatrix} \rho & x^1 & x^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad \rho \in \mathbb{R}^+, (x^1, x^2) \in \mathbb{R}^2.$$

(a) Find the left-invariant one-forms $\{\lambda^j, j = 0, 1, 2\}$ corresponding to a basis $\{X_j\}$ of $\mathfrak{g}$, and hence deduce that

$$h = \frac{1}{\rho^2}(d\rho^2 + (dx^1)^2 + (dx^2)^2)$$

is a left–invariant metric on G.

(b) Show that

$$d\lambda^i + \frac{1}{2} f^i_{jk} \lambda^j \wedge \lambda^k = 0,$$

where the constants f^i_{jk} should be determined.

(c) Find the left–invariant vector fields on G and show explicitly they generate a Lie algebra isomorphic to $\mathfrak{g}$.

2

Write an essay on topological degree of maps between manifolds.

3

Define a principal bundle (π, P, B, G).

Consider a connection

$$\omega = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma$$

on P, where A is a one–form on B and $\gamma \in G$.

(a) Show that ω does not depend on the choice of trivialisation of P if A transforms like a gauge potential on B.

(b) Use the right–invatiant vector fields on G to construct $\dim(B)$ linearly independent vector fields on P such that their contraction with ω vanishes. Show that these vector fields mutually comute iff $F = dA + A \wedge A = 0$.

4

Let (M, ω) be a symplectic manifold. Define a Hamiltonian vector field, and exibit a homeomorphis between the Lie algebras of functions on M with the Poisson bracket, and Hamiltonian vector fields with the Lie bracket. What is the kernel of this homomorphsm?

Consider a symplectic form ω on $M = S^2$ given by

$$\omega = i \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2},$$

where z is an affine coordinate on $\mathbb{CP}^1 = S^2$.

(a) Find a real vector field which generates a $U(1)$ action on S^2

$$z \longrightarrow e^{i\theta} z$$

where $\theta \in \mathbb{R}$.

(b) Show that this vector field is Hamiltonian, and find the corresponding Hamiltonian function.

END OF PAPER