UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 53

COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A spatially homogeneous and isotropic universe with expansion scale factor $a(t)$, containing matter with density ρ and pressure p, is described by the field equations

$$\begin{aligned}\frac{\dot{a}^2}{a^2} &= \frac{\kappa\rho}{3} - \frac{k}{a^2} \\ 0 &= \dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) \\ \frac{\ddot{a}}{a} &= -\frac{\kappa}{6}(\rho + 3p)\,,\end{aligned}$$

where $\kappa = 8\pi G$, the speed of light $c = 1$, k is the metric curvature parameter, and overdots denote differentiation with respect to the comoving proper time, t.

Define the particle horizon d_h, the Hubble parameter H, the deceleration parameter q and the density parameter Ω, and show that $q = \Omega$ for an isotropic and homogeneous universe containing blackbody radiation with zero cosmological constant.

In an open universe containing only blackbody radiation and with a zero cosmological constant, show that the proper distance to the particle horizon at the present time is given by

$$d_{h0} = \frac{1}{H_0\sqrt{1-\Omega_0}} \ln\left[\sqrt{\frac{(1-\Omega_0)}{\Omega_0}} + \sqrt{\left[1 + \frac{(1-\Omega_0)}{\Omega_0}\right]}\right]$$

where the subscript 0 denotes present-day values of quantities.

Describe the cosmological horizon problem and explain, without detailed calculation, how the phenomenon of inflation can resolve it.

2

It is proposed that a new type of 1keV stable neutrino and its antineutrino exist in equal numbers in the universe in addition to the three known neutrino types.

Explain how these new particle populations evolve in the early universe and calculate:

(i) the contribution that the new particles would make to the total density of the universe in units of the critical density;

(ii) their effect on the primordial abundance of helium-4 in the universe.

What conclusions can be drawn about this hypothetical new particle?

[You may assume that the present number density of photons in the universe is $n_{\gamma 0} = 410\text{cm}^{-3}$ and the present critical density is $\rho_{cr} = 5 \times 10^3\text{eV/cm}^3$. The weak interaction time is $t_{wk} = (1\text{MeV}/T)^5\text{s}$ and the temperature-time relation for the early universe is $t = g_*^{-1/2}(1\text{MeV}/T)^2\text{s}$ at temperature T, in MeV, where g_* is the total effective number of spin states at temperature T.]

3

Explain briefly why, for linear scalar perturbations on a flat, Friedmann-Robertson-Walker background, we can take the line element to be of the conformal Newtonian form

$$ds^2 = a^2(\eta)\left[(1+2\psi)d\eta^2 - (1-2\phi)\delta_{ij}dx^i dx^j\right] ,$$

where $a(\eta)$ is the scale factor as a function of conformal time η and the metric potentials ψ and ϕ are functions of η and the spatial coordinates x^i.

Under what conditions are ψ and ϕ equal?

Two metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ are related by a conformal transformation if $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ for some scalar function Ω^2. It can be shown that both metrics have the same null geodesics. By considering $\Omega^2 = a^{-2}(1+2\phi)$, show that the equation of affinely-parameterised geodesics, $d^2x^\mu/d\lambda^2 + \hat{\Gamma}^\mu_{\nu\rho}(dx^\nu/d\lambda)(dx^\rho/d\lambda) = 0$ (where $\hat{\Gamma}^\mu_{\nu\rho}$ is the connection for $\hat{g}_{\mu\nu}$), to first order in ψ and ϕ gives

$$\begin{aligned} \frac{d^2\eta}{d\lambda^2} + 2\frac{d\eta}{d\lambda}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} + 2\frac{dx^i}{d\lambda}\frac{d\eta}{d\lambda}\frac{\partial\Psi}{\partial x^i} &= 0\,, \\ \frac{d^2x^i}{d\lambda^2} + 2\left(\frac{d\eta}{d\lambda}\right)^2 \delta^{ij}\frac{\partial\Psi}{\partial x^j} &= 0\,, \end{aligned}$$

where $2\Psi \equiv \phi + \psi$ and the derivative $\mathrm{d}\Psi/\mathrm{d}\lambda$ is along the geodesic. Hence show that

$$\frac{d^2x^i}{d\eta^2} - 2\frac{dx^i}{d\eta}\left(\frac{\mathrm{d}\Psi}{\mathrm{d}\eta} + \frac{dx^j}{d\eta}\frac{\partial\Psi}{\partial x^j}\right) + 2\delta^{ij}\frac{\partial\Psi}{\partial x^j} = 0\,. \qquad (*)$$

For a photon received at $x^i = 0$ at time η_0 propagating along the spatial direction e^i (with $\delta_{ij}e^ie^j = 1$), so that $dx^i/d\eta\big|_{\eta_0} \propto e^i$, show that to first order

$$dx^i/d\eta\big|_{\eta_0} = (1+2\Psi)e^i\,.$$

By considering the solution of $(*)$ at zero order in Ψ, show that to first order

$$\frac{d^2x^i}{d\eta^2} - 2e^i\left(\frac{\mathrm{d}\Psi}{\mathrm{d}\eta} + e^j\frac{\partial\Psi}{\partial x^j}\right) + 2\delta^{ij}\frac{\partial\Psi}{\partial x^j} = 0\,.$$

Hence verify by substitution, or otherwise, that

$$x^i(\eta) = e^i(\eta - \eta_0) + 2e^i\int_{\eta_0}^{\eta}\Psi\, d\eta' - 2\int_{\eta_0}^{\eta}(\eta - \eta')(\delta^{ij} - e^ie^j)\partial_j\Psi\, d\eta'\,,$$

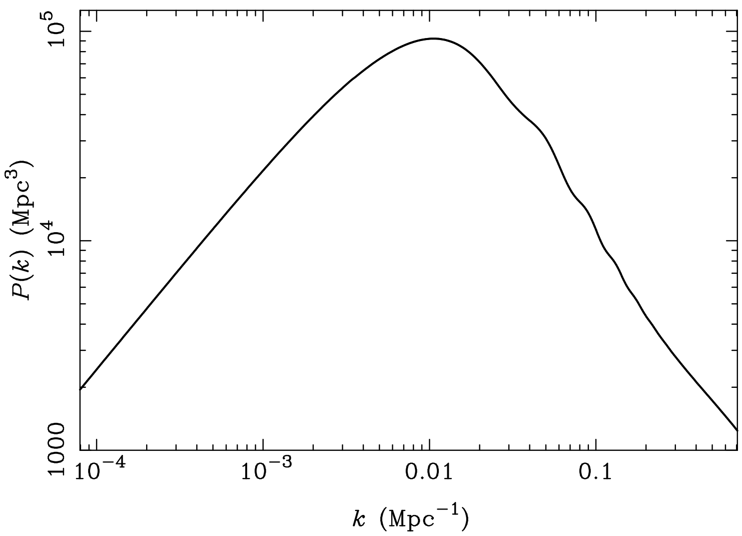
where the integrals are along the null geodesic.

If the photon leaves a source at time $\eta < \eta_0$, find the angular deflection of the apparent position of the source due to the metric perturbations when it is observed at η_0. Comment on observational consequences of this effect.

4 The *dimensional matter power spectrum* at conformal time η is defined by

$$\langle \Delta_m(\eta, \mathbf{k}) \Delta_m^*(\eta, \mathbf{k}') \rangle = P_{\Delta_m}(\eta; k) \delta(\mathbf{k} - \mathbf{k}') \,,$$

where $\Delta_m(\eta, \mathbf{k})$ is the fractional matter overdensity in the comoving gauge in Fourier space. The figure below shows $P_{\Delta_m}(\eta; k)$ at redshift zero calculated in linear perturbation theory for an inflationary model with potential $V(\Phi) = V_0(\Phi/M_{\rm Pl})^2$, where $M_{\rm Pl}$ is the reduced Planck mass ($8\pi G = 1/M_{\rm Pl}^2$) and V_0 is a constant.



Explain the main features of this matter power spectrum and discuss quantitatively the asymptotic behaviour for small and large k.

Describe how the following changes would affect the matter power spectrum at redshift zero. (For parts 2 and 3 you should provide quantitative details.)

1. The energy density in dark energy is increased keeping the energy density in matter and radiation fixed and preserving spatial flatness.
2. The constant V_0 is doubled.
3. The inflationary potential is changed to a quartic potential $V \propto \Phi^4$.

Assuming fluctuations at scale $k = 10^{-2}\,\mathrm{Mpc}^{-1}$ were generated $N \sim 60$ e-folds before the end of inflation, estimate the range of field values $\Delta\Phi/M_{\rm Pl}$ that are probed by the range $10^{-3}\,\mathrm{Mpc}^{-1} \leqslant k \leqslant 10^{-1}\,\mathrm{Mpc}^{-1}$ (i.e. $\Delta \ln k \approx 5$) for $V = V_0(\Phi/M_{\rm Pl})^2$.

[You may assume that in slow-roll inflation, the dimensionless power spectrum of primordial curvature perturbations is

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{8}{3} \left(\frac{V^{1/4}}{\sqrt{8\pi} M_{\rm Pl}} \right)^4 \frac{1}{\epsilon_V} \,,$$

which has spectral index $n_s - 1 = 2\eta_V - 6\epsilon_V$ in terms of the slow-roll parameters $\epsilon_V \equiv \frac{1}{2} M_{\rm Pl}^2 (V'/V)^2$ and $\eta_V \equiv M_{\rm Pl}^2 (V''/V)$, where primes denote differentiation with respect to Φ.]

END OF PAPER