UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 1:30 pm to 4:30 pm

PAPER 52

THE STANDARD MODEL

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Under the parity transformation $x \mapsto x_p = (x^0, -\mathbf{x})$, a Dirac fermion ψ transforms as

$$\psi(x) \mapsto \eta_P \psi^P(x), \quad \text{with} \quad \psi^P(x) = \gamma^0 \psi(x_P)\,.$$

Show that the γ^0 is necessary in order to have a parity-invariant theory.

Under charge conjugation

$$\psi(x) \mapsto \eta_C \psi^c(x), \quad \text{with} \quad \psi^c(x) = C\bar{\psi}^T(x)$$

where T denotes transpose. What condition on C is implied by requiring charge conjugation symmetry?

Let ψ represent an up quark, which interacts with the gluon field A^a_μ ($a = 1, \dots, 8$). Denoting the $SU(3)$ generators in the fundamental representation as T^a, the gauge-covariant derivative acting on ψ can be written as $D_\mu = \partial_\mu + \mathrm{i}gA^a_\mu T^a$. Given that the quark-gluon interaction term is CP-invariant, derive an expression for how $A^a_\mu T^a$ transforms under CP.

The gluon field strength may be written as

$$F^a_{\mu\nu} T^a = \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a + \mathrm{i}g[A_\mu T^b, A_\nu T^c]\,.$$

Calculate the CP transformation of $F^a_{\mu\nu} T^a$ and determine whether the following term conserves CP:

$$\mathcal{L}_\theta(x) = \theta\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)\,,$$

where θ is a real constant.

2

Consider two theories. The first involves only a 3-component real scalar field $\Phi(x)$ with Lagrangian

$$\mathcal{L}_1 = \frac{1}{2}\partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{1}{2} m^2 \Phi \cdot \Phi - \frac{\lambda}{4}(\Phi \cdot \Phi)^2\,.$$

The second theory includes an interaction with a gauge field B^a_μ ($a = 1, \dots, 3$) through the gauge-covariant derivative $D_\mu = \partial_\mu + \mathrm{i}gt^a B^a_\mu$:

$$\mathcal{L}_2 = \frac{1}{2} D_\mu \Phi \cdot D^\mu \Phi - \frac{1}{2} m^2 \Phi \cdot \Phi - \frac{\lambda}{4}(\Phi \cdot \Phi)^2 - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}\,.$$

$F^a_{\mu\nu}$ is the field strength associated with B^a_μ.

Write an essay discussing the consequences of spontaneous symmetry breaking in each of these two models. As part of the essay, identify the symmetries of the Lagrangians and assume that one component of Φ has a nonzero vacuum expectation value. Be sure to identify the physical degrees-of-freedom via calculation in each of the two cases.

3

Given a lepton ℓ and a neutrino ν which form a left-handed weak $SU(2)$ doublet $L(x) = \begin{pmatrix} \nu(x) \\ \ell(x) \end{pmatrix}_L = \frac{1}{2}(1-\gamma^5)\begin{pmatrix} \nu(x) \\ \ell(x) \end{pmatrix}$ and a right-handed weak $SU(2)$ singlet $R(x) = \ell_R(x) = \frac{1}{2}(1+\gamma^5)\ell(x)$, the coupling of $L(x)$ or $R(x)$ to the neutral vector gauge boson field $Z_\mu(x)$ is given by the interaction Lagrangian

$$\mathcal{L}_I = \frac{g}{\cos\theta_W} J_Z^\mu Z_\mu \,.$$

The weak neutral current is

$$J_Z^\mu = \bar{\Psi}\gamma^\mu(T^3 - Q\sin^2\theta_W)\Psi$$

with $\Psi(x) = L(x)$ or $R(x)$; T^3 is one of the generators of $SU(2)$ in the appropriate representation, and Q is the electromagnetic charge matrix. By considering the results of T^3 and Q acting on $L(x)$ and $R(x)$, show that the

$$J_Z^\mu = \sum_j \bar{\psi}^j \gamma^\mu (c_V^j - c_A^j\gamma^5)\psi^j$$

and determine c_V^j and c_A^j for the 2 cases $\psi^j(x) = \ell(x)$ and $\nu(x)$.

Assume $\langle 0|Z_\mu(0)|Z(p,\varepsilon)\rangle = \varepsilon_\mu(p)$ and

$$\sum_{Z \text{ spins}} \varepsilon_\mu(p)\varepsilon_\nu^*(p) = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\,.$$

Find the widths for $\Gamma(Z \to \ell\bar{\ell})$ and $\Gamma(Z \to \nu\bar{\nu})$, assuming ℓ and ν can be treated as massless.

[You may use

$$\Gamma = \frac{1}{2m_Z}\prod_f \left[\int \frac{\mathrm{d}^3k_f}{(2\pi)^3 2k_f^0}\right](2\pi)^4\delta^{(4)}\left(p - \sum_f k_f\right)\frac{1}{N_s}\sum_{\text{spins}}|\mathcal{M}|^2$$

where f labels the final state momenta (here $f = 1,2$) and N_s is the number of spin degrees-of-freedom of the Z. Also note that $\mathrm{Tr}\,\gamma^\alpha\gamma^\beta\gamma^\tau\gamma^\delta = 4(\eta^{\alpha\beta}\eta^{\tau\delta} - \eta^{\alpha\tau}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\tau})$ and $\mathrm{Tr}\,\gamma^\alpha\gamma^\beta\gamma^\tau\gamma^\delta\gamma^5 = 4\mathrm{i}\epsilon^{\alpha\beta\tau\delta}$.]

4

Consider the neutral kaons that have definite strangeness S, the K^0 with $S = 1$ and the $\bar{K}^0$ with $S = -1$. Draw a one-loop Standard Model Feynman diagram contributing to the mixing of K^0 and $\bar{K}^0$.

Let H' represent the weak Hamiltonian correct to at least 2nd order in perturbation theory. Let us represent the matrix elements of H' as

$$\begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\bar{K}^0\rangle \\ \langle \bar{K}^0|H'|K^0\rangle & \langle \bar{K}^0|H'|\bar{K}^0\rangle \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} .$$

Derive a relationship between matrix elements assuming CPT-invariance.

If CP were respected by the weak interactions, what condition on the matrix elements R_{12} and R_{21} would hold?

Assuming CPT-invariance, obtain the mass eigenstates K_S^0 and K_L^0 in terms of the weak eigenstates K^0 and $\bar{K}^0$ at the time of their production, i.e. consider only the Hermitian part of R. [You may take it as an experimental fact that the K_L^0 is more massive than the K_S^0.]

Let $|K_1^0\rangle$ and $|K_2^0\rangle$ respectively label the states $|K_S^0\rangle$ and $|K_L^0\rangle$ in the hypothetical limit where CP is conserved. How are $|K_1^0\rangle$ and $|K_2^0\rangle$ related to $|K^0\rangle$ and $|\bar{K}^0\rangle$?

Choosing the phase convention so that $|K_1^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)$, show that

$$\begin{aligned} |K_S^0\rangle &= \sqrt{\frac{1}{1+|\epsilon|^2}}\left(|K_1^0\rangle + \epsilon|K_2^0\rangle\right), \\ |K_L^0\rangle &= \sqrt{\frac{1}{1+|\epsilon|^2}}\left(|K_2^0\rangle + \epsilon|K_1^0\rangle\right), \end{aligned}$$

with the definition

$$\epsilon = \frac{\sqrt{R_{12}} - \sqrt{R_{21}}}{\sqrt{R_{12}} + \sqrt{R_{21}}} .$$

END OF PAPER