UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 9:00 am to 12:00 pm

PAPER 50

ADVANCED QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Assuming

$$\langle q'|e^{-i\frac{1}{2}\hat{p}^2 T}|q\rangle = \left(\frac{1}{2\pi i T}\right)^{\frac{1}{2}} e^{i\frac{(q'-q)^2}{2T}} \,,$$

where $\hat{q}, \hat{p}$ are the usual quantum mechanical position and momentum operators with $\hat{q}|q\rangle = q|q\rangle$, obtain the path integral representation involving a functional integral over paths $q(t)$ with $q(0) = q\,,\ q(T) = q'$,

$$\langle q'|e^{-i\hat{H}T}|q\rangle = \int \mathrm{d}[q]\, e^{iS[q]} \,,$$

where the Hamiltonian $\hat{H}$ and the action $S[q]$ are given by

$$\hat{H} = \tfrac{1}{2}\,\hat{p}^2 + V(\hat{q})\,, \qquad S[q] = \int_0^T \mathrm{d}t\, (\tfrac{1}{2}\,\dot{q}^2 - V(q))\,.$$

For $V(q)$ quadratic in q, show that

$$\langle q'|e^{-i\hat{H}T}|q\rangle = N\, e^{iS[q_c]} \,,$$

where q_c is the classical path linking q and q' and N is independent of q, q'. Show that N can be expressed in terms of the determinant of the operator $\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(q_c)$ acting on functions $f(t)$ with $f(0) = f(T) = 0$.

Consider the action

$$S[q] = \int_{-\infty}^{\infty} \mathrm{d}t\, (\tfrac{1}{2}\,\dot{q}^2 - \tfrac{1}{2}m^2 q^2 + Jq)\,,$$

for arbitrary $J(t)$. Show that the functional integral over appropriate paths $q(t)$ with $-\infty < t < \infty$ can in this case be expressed in the form

$$\int \mathrm{d}[q]\, e^{iS[q]} = N\, e^{-\frac{1}{2}i\int \mathrm{d}t\mathrm{d}t'\, J(t)G(t-t')J(t')} \,, \quad G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega t}\, \frac{1}{\omega^2 - m^2 + i\epsilon} \,.$$

2

For the Lagrangian, in d-dimensions,

$$\mathcal{L} = -\tfrac{1}{2}\,(\partial\phi)^2 - \tfrac{1}{24}\lambda\,\phi^4\,,$$

what are the momentum space Feynman rules?

Let $\hat{\tau}_n(p_1,\ldots,p_n)$, $\sum_i p_i = 0$, be the amplitude corresponding to connected one particle irreducible graphs with n external lines after factoring off $i(2\pi)^d\delta^d(\sum_i p_i)$. Disregarding sub-divergencies demonstrate that the integrals for $\hat{\tau}_n$ are divergent when $d=4$ only for restricted n.

What are the results for $\hat{\tau}_2, \hat{\tau}_4$ at lowest order (no loops)? Draw the one loop graphs which are relevant for $\hat{\tau}_2, \hat{\tau}_4$. Write the one loop contribution to $\hat{\tau}_4$ in terms of the integral

$$f(P^2) = \frac{1}{(2\pi)^d i}\int \mathrm{d}^d k\,\frac{1}{(k^2 - i\epsilon)((P-k)^2 - i\epsilon)}\,.$$

Under analytic continuation to a Euclidean metric $\mathrm{d}^d k \to i\,\mathrm{d}^d k$. Show that the divergence of the integral when $d=4$ is represented by a pole as $d \to 4$ of the form

$$f(P^2) \sim \frac{1}{16\pi^2}\,\frac{2}{4-d}\,.$$

Show how a finite result for $\hat{\tau}_4$ when $d=4$ may be achieved.

[*It may be useful to recall the properties of the Gamma function*

$$\Gamma(x) = \int_0^\infty \mathrm{d}\lambda\,\lambda^{x-1}\,e^{-\lambda} \ \text{ if } x>0\,, \quad \Gamma(x+1) = x\Gamma(x)\,, \quad \Gamma(1) = 1\,. \quad]$$

3

Consider a renormalisable quantum field theory with a single dimensionless coupling g and no mass parameters. Let $\langle\phi(x_1)\ldots\phi(x_n)\rangle$ be the finite correlation function for scalar fields ϕ determined by perturbation expansion of the quantum field theory as a series in g. Why must this also depend on an additional scale μ? Describe the derivation of the RG equation

$$(\mathcal{D}+n\gamma(g))\langle\phi(x_1)\ldots\phi(x_n)\rangle = 0\,, \quad \mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\,,$$

and briefly discuss its interpretation. What are UV and IR fixed points?

Assuming

$$\int \mathrm{d}^4x\, e^{ip\cdot x}\langle\phi(x)\phi(0)\rangle = -i\,\frac{d(p^2/\mu^2,g)}{p^2}\,,$$

show how the behaviour of $d(p^2/\mu^2,g)$ for large p^2 depends on the form of $\beta(g)$. If $\beta(g) = -bg^3$, $\gamma(g) = cg^2$ with $b>0$, find an expression for $d(p^2/\mu^2)$ for large p^2. If $\beta(g) = -bg^3 - ag^5$ and $b<0,\ a>0$, what happens for large p^2?

For an $SU(N)$ gauge theory with n fermions belonging to the N-dimensional representation and a gauge coupling g then in the RG equation we may take with $a = g^2/16\pi^2$

$$\mathcal{D} = \mu^2\frac{\partial}{\partial\mu^2} + \beta(a)\frac{\partial}{\partial a}\,, \quad \beta(a) = -\beta_0\,a^2 - \beta_1\,a^3 + \mathrm{O}(a^4)\,,$$
$$\beta_0 = \tfrac{11}{3}N - \tfrac{2}{3}n\,, \quad \beta_1 = \tfrac{34}{3}N^2 - (\tfrac{13}{3}N - \tfrac{1}{N})n\,.$$

Explain why the running coupling for large μ may be written, if n is not too large, as

$$a(\mu^2) = \frac{1}{\beta_0 \ln\frac{\mu^2}{\Lambda^2}}\,,$$

for some suitable Λ. Show that it is possible to choose n so that $0<\beta_0 \ll N$ and $\beta_1<0$ so that there is an IR fixed point at which a is small.

4

Let $A_{\mu a}(x)$, $a = 1, 2, 3$, be a gauge field for an $SU(2)$ gauge group. Under an infinitesimal gauge transformation $\delta A_{\mu a} = \partial_\mu \lambda_a + g\, \varepsilon_{abc} A_{\mu b} \lambda_c \equiv (D_\mu \lambda)_a$, how does

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g\, \varepsilon_{abc} A_{\mu b} A_{\nu c}$$

transform?

Explain why

$$\mathcal{L} = -\tfrac{1}{4}\, F^{\mu\nu}{}_a F_{\mu\nu a}$$

is gauge invariant. If the action is $S = \int \mathrm{d}^d x\, \mathcal{L}$, why do the classical equations of motion not fully determine the time evolution of $A_{\mu a}(x)$? Briefly explain why just using S as the action is not satisfactory for a quantum field theory.

Introducing anti-commuting fields $c_a(x), \bar{c}_a(x)$ and a real field $b_a(x)$, define an operation s by

$$sA_{\mu a} = (D_\mu c)_a\,, \quad sc_a = -\tfrac{1}{2} g\, \varepsilon_{abc} c_b c_c\,, \quad s\bar{c}_a = -b_a\,, \quad sb_a = 0\,,$$

where $s(XY) = sX\, Y \pm X\, sY$, with $-$ if X is anti-commuting. Verify that $s^2 = 0$.

Show that

$$\mathcal{L}_q = -\tfrac{1}{4}\, F^{\mu\nu}{}_a F_{\mu\nu a} - s(\partial^\mu \bar{c}_a\, A_{\mu a} + \tfrac{1}{2} \xi\, \bar{c}_a\, b_a)\,,$$

satisfies $s\mathcal{L}_q = 0$. Explain why for $S_q = \int \mathrm{d}^d x\, \mathcal{L}_q$ a perturbative expansion in g of the functional integral

$$\langle X \rangle = \int \mathrm{d}[A, c, \bar{c}, b]\ X\, e^{iS_q} \Big/ \int \mathrm{d}[A, c, \bar{c}, b]\ e^{iS_q}\,,$$

may be obtained for any X constructed from $A_{\mu a}, c_a, \bar{c}_a . b_a$. If $\xi = 1$ and $g = 0$ what is the propagator for $A_{\mu a}$?

Assume that $\langle sY \rangle = 0$ for any Y. If X is gauge invariant and is independent of ξ, show that

$$\frac{\partial}{\partial \xi} \langle X \rangle = 0\,.$$

END OF PAPER