You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
Assuming
\[ \langle q' | e^{-i \hat{H} T} | q \rangle = \left( \frac{1}{2\pi i T} \right)^{\frac{1}{2}} e^{i \frac{\omega^2 \hat{q}^2}{2T}}, \]
where \( \hat{q}, \hat{p} \) are the usual quantum mechanical position and momentum operators with \( \hat{q}|q\rangle = q|q\rangle \), obtain the path integral representation involving a functional integral over paths \( q(t) \) with \( q(0) = q \), \( q(T) = q' \),
\[ \langle q' | e^{-i \hat{H} T} | q \rangle = \int d[q] e^{iS[q]}, \]
where the Hamiltonian \( \hat{H} \) and the action \( S[q] \) are given by
\[ \hat{H} = \frac{1}{2} \hat{p}^2 + V(\hat{q}), \quad S[q] = \int_0^T dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right). \]

For \( V(q) \) quadratic in \( q \), show that
\[ \langle q' | e^{-i \hat{H} T} | q \rangle = N e^{iS[q_c]}, \]
where \( q_c \) is the classical path linking \( q \) and \( q' \) and \( N \) is independent of \( q, q' \). Show that \( N \) can be expressed in terms of the determinant of the operator \( \frac{d^2}{dt^2} + V''(q_c) \) acting on functions \( f(t) \) with \( f(0) = f(T) = 0 \).

Consider the action
\[ S[q] = \int_{-\infty}^{\infty} dt \left( \frac{1}{2} \dot{q}^2 - \frac{1}{4} m^2 q^2 + Jq \right), \]
for arbitrary \( J(t) \). Show that the functional integral over appropriate paths \( q(t) \) with \( -\infty < t < \infty \) can in this case be expressed in the form
\[ \int d[q] e^{iS[q]} = N e^{-\frac{1}{4} \int dtd't' J(t) G(t-t') J(t')}, \quad G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \frac{1}{\omega^2 - m^2 + i\epsilon}. \]
For the Lagrangian, in $d$-dimensions,

$$L = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{24} \lambda \phi^4,$$

what are the momentum space Feynman rules?

Let $\hat{\tau}_n(p_1, \ldots, p_n), \sum_i p_i = 0$, be the amplitude corresponding to connected one particle irreducible graphs with $n$ external lines after factoring off $i(2\pi)^d \delta^d(\sum_i p_i)$. Disregarding sub-divergencies demonstrate that the integrals for $\hat{\tau}_n$ are divergent when $d = 4$ only for restricted $n$.

What are the results for $\hat{\tau}_2, \hat{\tau}_4$ at lowest order (no loops)? Draw the one loop graphs which are relevant for $\hat{\tau}_2, \hat{\tau}_4$. Write the one loop contribution to $\hat{\tau}_4$ in terms of the integral

$$f(P^2) = \frac{1}{(2\pi)^d i} \int d^d k \frac{1}{(k^2 - i\epsilon)((P - k)^2 - i\epsilon)}.$$

Under analytic continuation to a Euclidean metric $d^d k \rightarrow i d^d k$. Show that the divergence of the integral when $d = 4$ is represented by a pole as $d \rightarrow 4$ of the form

$$f(P^2) \sim \frac{1}{16\pi^2} \frac{2}{4 - d}.$$

Show how a finite result for $\hat{\tau}_4$ when $d = 4$ may be achieved.

[It may be useful to recall the properties of the Gamma function

$$\Gamma(x) = \int_0^\infty d\lambda \lambda^{x-1} e^{-\lambda} \text{ if } x > 0, \quad \Gamma(x + 1) = x\Gamma(x), \quad \Gamma(1) = 1.$$]
Consider a renormalisable quantum field theory with a single dimensionless coupling $g$ and no mass parameters. Let $\langle \phi(x_1) \ldots \phi(x_n) \rangle$ be the finite correlation function for scalar fields $\phi$ determined by perturbation expansion of the quantum field theory as a series in $g$. Why must this also depend on an additional scale $\mu$? Describe the derivation of the RG equation

$$(\mathcal{D} + n \gamma(g)) \langle \phi(x_1) \ldots \phi(x_n) \rangle = 0, \quad \mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g},$$

and briefly discuss its interpretation. What are UV and IR fixed points?

Assuming

$$\int d^4x e^{ip \cdot x} \langle \phi(x) \phi(0) \rangle = -i \frac{d(p^2/\mu^2, g)}{p^2},$$

show how the behaviour of $d(p^2/\mu^2, g)$ for large $p^2$ depends on the form of $\beta(g)$. If $\beta(g) = -bg^3$, $\gamma(g) = cg^2$ with $b > 0$, find an expression for $d(p^2/\mu^2)$ for large $p^2$. If $\beta(g) = -bg^3 - ag^5$ and $b < 0$, $a > 0$, what happens for large $p^2$?

For an $SU(N)$ gauge theory with $n$ fermions belonging to the $N$-dimensional representation and a gauge coupling $g$ then in the RG equation we may take with $a = g^2/16\pi^2$

$${\mathcal D} = \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a) \frac{\partial}{\partial a}, \quad \beta(a) = -\beta_0 a^2 - \beta_1 a^3 + O(a^4), \quad \beta_0 = \frac{11}{3} N - \frac{2}{3} n, \quad \beta_1 = \frac{34}{3} N^2 - (\frac{13}{3} N - \frac{1}{3}) n.$$

Explain why the running coupling for large $\mu$ may be written, if $n$ is not too large, as

$$a(\mu^2) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}},$$

for some suitable $\Lambda$. Show that it is possible to choose $n$ so that $0 < \beta_0 \ll N$ and $\beta_1 < 0$ so that there is an IR fixed point at which $a$ is small.
Let $A_{\mu a}(x)$, $a = 1, 2, 3$, be a gauge field for an $SU(2)$ gauge group. Under an infinitesimal gauge transformation $\delta A_{\mu a} = \partial_\mu \lambda_a + g \varepsilon_{abc} A_{\mu b} \lambda_c \equiv (D_\mu \lambda)_a$, how does

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g \varepsilon_{abc} A_{\mu b} A_{\nu c}$$

transform?

Explain why

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu}$$

is gauge invariant. If the action is $S = \int d^4 x \mathcal{L}$, why do the classical equations of motion not fully determine the time evolution of $A_{\mu a}(x)$? Briefly explain why just using $S$ as the action is not satisfactory for a quantum field theory.

Introducing anti-commuting fields $c_a(x), \bar{c}_a(x)$ and a real field $b_a(x)$, define an operation $s$ by

$$s A_{\mu a} = (D_\mu c)_a, \quad sc_a = -\frac{1}{2} g \varepsilon_{abc} c_b c_c, \quad s \bar{c}_a = -b_a, \quad sb_a = 0,$$

where $s(XY) = sXY \pm XSY$, with $-$ if $X$ is anti-commuting. Verify that $s^2 = 0$.

Show that

$$\mathcal{L}_q = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - s(\partial_\mu \bar{c}_a A_{\mu a} + \frac{1}{2} \xi \bar{c}_a b_a),$$

satisfies $s\mathcal{L}_q = 0$. Explain why for $S_q = \int d^4 x \mathcal{L}_q$ a perturbative expansion in $g$ of the functional integral

$$\langle X \rangle = \int d[A, c, \bar{c}, b] X e^{iS_q} / \int d[A, c, \bar{c}, b] e^{iS_q},$$

may be obtained for any $X$ constructed from $A_{\mu a}, c_a, \bar{c}_a, b_a$. If $\xi = 1$ and $g = 0$ what is the propagator for $A_{\mu a}$?

Assume that $\langle sY \rangle = 0$ for any $Y$. If $X$ is gauge invariant and is independent of $\xi$, show that

$$\frac{\partial}{\partial \xi} \langle X \rangle = 0.$$