PAPER 5

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt no more than **THREE** questions.

There are **FOUR** questions in total.

The questions carry equal weight.

**STATIONERY REQUIREMENTS**

- Cover sheet
- Treasury Tag
- Script paper

**SPECIAL REQUIREMENTS**

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
1

What is a Fréchet space? Describe without proof how the topology of a Fréchet space can be defined in terms of seminorms.

What is the weak topology on a Fréchet space? Show that it is a Hausdorff topology. Show that a linear functional on a Fréchet space is continuous in the weak topology if and only if it is continuous in the original topology.

Suppose that $T$ is a linear mapping from a Fréchet space $(E_1, \tau_1)$ to a Fréchet space $(E_2, \tau_2)$. Show that $T$ is continuous when $E_1$ and $E_2$ are given their original topologies if and only if it is continuous when they are given their weak topologies.

Let $\Omega$ denote the Bernoulli space $\prod_{j=1}^{\infty} \{0,1\}_j$, with its product topology. What is a cylinder set in $\Omega$? Show that a cylinder set is open and closed, and that the cylinder sets are a base for the topology on $\Omega$.

If $A$ is a subset of $\Omega$, let $I_A$ denote its indicator function. Show that $S(\Omega) = \text{span}\{I_C : C \text{ a cylinder set}\}$ is a dense linear subspace of $(C_R(\Omega), \|\cdot\|_{\infty})$.

Suppose that $\phi$ is a positive linear functional on $C_R(\Omega)$, with $\phi(I_\Omega) = 1$. Show that $\phi$ is continuous.

If $C$ is a cylinder set, let $l(C) = \phi(I_C)$. Show that $\phi$ extends to a unique countably additive function on the open subsets of $\Omega$.

Describe briefly how to show that there exists a unique probability measure $P$ on $\Omega$ such that $\phi(f) = \int_{\Omega} f \, dP$, for each $f \in C_R(\Omega)$.

Suppose that $(X, d)$ is a compact metric space and that $\phi$ is a positive linear functional on $C_R(X)$, with $\phi(I_X) = 1$. Show that there exists a probability measure $Q$ on $X$ such that $\phi(f) = \int_X f \, dQ$, for each $f \in C_R(X)$. [You may assume that there is a continuous surjection $h$ of $\Omega$ onto $X$.]

3

Suppose $\mu$ and $\nu$ are finite non-negative measures on a measurable space $(\Omega, \Sigma)$. Show that there exist a non-negative $f \in L^1(\mu)$ and a set $B \in \Sigma$ with $\mu(B) = 0$ such that $\nu(A) = \int_A f \, d\mu + \nu(A \cap B)$ for each $A \in \Sigma$.

What does it mean to say that $\nu$ is absolutely continuous with respect to $\mu$? Show that $\nu$ is absolutely continuous with respect to $\mu$ if and only if there exist a non-negative $f \in L^1(\mu)$ such that $\nu(A) = \int_A f \, d\mu$ for each $A \in \Sigma$.

Suppose that $f \in L^1(\Omega, \Sigma, \mu)$ and that $\Sigma_0$ is a sub-$\sigma$-field of $\Sigma$. Show that there exists $f_0 \in L^1(\Omega, \Sigma_0, \mu)$ such that $\int_A f \, d\mu = \int_A f_0 \, d\mu$ for all $A \in \Sigma_0$. 

Part III, Paper 5
Let $\lambda_d$ denote Lebesgue measure on the Borel sets of $\mathbb{R}^d$. Suppose that $\mathcal{G}$ is a collection of open balls in $\mathbb{R}^d$ which cover a compact set $K$. Show that there exists a finite subset $\mathcal{F}$ of $\mathcal{G}$ whose elements are disjoint, such that $\lambda_d(\bigcup_{U \in \mathcal{F}} U) \geq \lambda_d(K)/3^d$.

Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}^d$. What does it mean to say that $\mu$ has a spherical derivative at $x$? Suppose that $\mu$ and $\lambda_d$ are mutually singular. Show that $\mu$ has spherical derivative 0 at $\lambda_d$-almost every point of $\mathbb{R}^d$.

Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}^1$ and that $\mu$ and $\lambda_1$ are mutually singular. Show that the cumulative distribution $F$ of $\mu$ (where $F(t) = \mu((-\infty, t])$ ) is differentiable, with derivative 0, at $\lambda_1$-almost every point of $\mathbb{R}$.

END OF PAPER