UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 5

INTRODUCTION TO FUNCTIONAL ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is a Fréchet space? Describe without proof how the topology of a Fréchet space can be defined in terms of seminorms.

What is the weak topology on a Fréchet space? Show that it is a Hausdorff topology. Show that a linear functional on a Fréchet space is continuous in the weak topology if and only if it is continuous in the original topology.

Suppose that T is a linear mapping from a Fréchet space (E_1, τ_1) to a Fréchet space (E_2, τ_2). Show that T is continuous when E_1 and E_2 are given their original topologies if and only if it is continuous when they are given their weak topologies.

2

Let Ω denote the Bernoulli space $\prod_{j=1}^{\infty}\{0,1\}_j$, with its product topology. What is a *cylinder set* in Ω? Show that a cylinder set is open and closed, and that the cylinder sets are a base for the topology on Ω.

If A is a subset of Ω, let I_A denote its indicator function, Show that $S(\Omega) = \text{span}\{I_C : C \text{ a cylinder set}\}$ is a dense linear subspace of $(C_{\mathbf{R}}(\Omega), \|.\|_\infty)$.

Suppose that ϕ is a positive linear functional on $C_{\mathbf{R}}(\Omega)$, with $\phi(I_\Omega) = 1$. Show that ϕ is continuous.

If C is a cylinder set, let $l(C) = \phi(I_C)$. Show that ϕ extends to a unique countably additive function on the open subsets of Ω.

Describe briefly how to show that there exists a unique probability measure $\mathbf{P}$ on Ω such that $\phi(f) = \int_\Omega f\, d\mathbf{P}$, for each $f \in C_{\mathbf{R}}(\Omega)$.

Suppose that (X, d) is a compact metric space and that ϕ is a positive linear functional on $C_{\mathbf{R}}(X)$, with $\phi(I_X) = 1$. Show that there exists a probability measure $\mathbf{Q}$ on X such that $\phi(f) = \int_X f\, d\mathbf{Q}$, for each $f \in C_{\mathbf{R}}(X)$. [You may assume that there is a continuous surjection h of Ω onto X.]

3

Suppose μ and ν are finite non-negative measures on a measurable space (Ω, Σ). Show that there exist a non-negative $f \in L^1(\mu)$ and a set $B \in \Sigma$ with $\mu(B) = 0$ such that $\nu(A) = \int_A f\, d\mu + \nu(A \cap B)$ for each $A \in \Sigma$.

What does it mean to say that ν is *absolutely continuous* with respect to μ? Show that ν is absolutely continuous with respect to μ if and only if there exist a non-negative $f \in L^1(\mu)$ such that $\nu(A) = \int_A f\, d\mu$ for each $A \in \Sigma$.

Suppose that $f \in L^1(\Omega, \Sigma, \mu)$ and that Σ_0 is a sub-σ-field of Σ. Show that there exists $f_0 \in L^1(\Omega, \Sigma_0, \mu)$ such that $\int_A f\, d\mu = \int_A f_0\, d\mu$ for all $A \in \Sigma_0$.

4

Let λ_d denote Lebesgue measure on the Borel sets of $\mathbf{R}^d$. Suppose that $\mathcal{G}$ is a collection of open balls in $\mathbf{R}^d$ which cover a compact set K. Show that there exists a finite subset $\mathcal{F}$ of $\mathcal{G}$ whose elements are disjoint, such that $\lambda_d(\cup_{U \in \mathcal{F}} U) \geqslant \lambda_d(K)/3^d$.

Suppose that μ is a Borel probability measure on $\mathbf{R}^d$. What does it mean to say that μ has a spherical derivative at x? Suppose that μ and λ_d are mutually singular. Show that μ has spherical derivative 0 at λ_d-almost every point of $\mathbf{R}^d$.

Suppose that μ is a Borel probability measure on $\mathbf{R}^1$ and that μ and λ_1 are mutually singular. Show that the cumulative distribution F of μ (where $F(t) = \mu((-\infty, t])$) is differentiable, with derivative 0, at λ_1-almost every point of $\mathbf{R}$.

END OF PAPER