PAPER 49

STATISTICAL FIELD THEORY

Attempt no more than TWO questions.
There are THREE questions in total.
The questions carry equal weight.
Give an account of the Landau-Ginsberg (LG) theory of phase transitions in the context of a scalar field theory which should include a discussion of the following points:

(i) the idea of an order parameter;
(ii) the distinction between first-order and continuous phase transitions and how their defining properties are explained;
(iii) the idea of universality giving an example;
(iv) the idea of critical exponents and how they may be derived;
(v) the reason why a line of first order transitions must terminate in a critical point associated with a continuous phase transition;
(vi) the explanation of the features of a two-dimensional phase diagram containing a tricritical point in which you should identify a line of three-phase coexistence and give a reason for its occurrence.

Near to a continuous phase transition at $T = T_C$, the critical exponents $\alpha$, $\beta$, $\gamma$ and $\delta$ are defined by

Specific heat: $C_V \sim |t|^{-\alpha}$ ($h = 0$),
Magnetization: $M \sim |t|^{\beta}$ ($t < 0, h = 0$),
Susceptibility: $\chi \sim |t|^{-\gamma}$ ($h = 0$),
Magnetization: $M \sim |h|^{1/\delta}$ ($t = 0$),

where $t = (T - T_C)/T_C$ and $h$ is the applied magnetic field. Calculate $\alpha$, $\beta$, $\gamma$ and $\delta$ for a tricritical point and verify the scaling relations

$\alpha + 2\beta + \gamma = 2, \quad \beta\delta = \beta + \gamma$. 

Part III, Paper 49
A spin model in $D$ dimensions is defined on a cubic lattice of spacing $a$ with $N$ sites and with spin $\sigma_n$ on the $n$-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$H(u, \sigma) = \sum_i u_i O_i(\{\sigma\}),$$

where the $u_i$ are coupling constants with $u = (u_1, u_2, \ldots)$. In particular, $H$ contains the term $-h \sum_n \sigma_n$ where $h$ is the magnetic field. The partition function is given by

$$Z(u, C, N) = \sum_{\sigma} \exp(-\beta H(u, \sigma) - \beta NC).$$

Define the two-point correlation function $G(r)$ for the theory and state how the correlation length $\xi$ parametrizes its behaviour as $|r| \to \infty$. State how the susceptibility $\chi$ can be expressed in terms of $G(r)$.

Explain how the renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after $p$ iterations, yields a blocked partition function $Z(u_p, C_p, N_p)$ which predicts the same large-scale properties for the system as does $Z(u, C, N)$. State how $a$ and $N$ rescale in terms of the RG scale factor $b$.

Derive the RG equation for the free energy $F(u_p, C_p)$, and explain how it may be expressed in terms of a singular part, $f(u)$, which obeys the RG equation

$$f(u_0) = b^{-pD} f(u_p) + \sum_{j=0}^{p-1} b^{-jD} g(u_j), \quad p > 0.$$ 

What is the origin of the function $g(u)$ which determines the inhomogeneous part of the transformation?

Explain the idea of a fixed point, relevant and irrelevant operators, a critical surface and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived. In the case where there are two relevant couplings $t = (T - T_c)/T_c$ and $h$, and stating any assumptions you make, derive the scaling hypothesis for the free energy near to a critical point:

$$F(u_0, C_0) = |t|^{D/\lambda_t} \left( f_\pm \left( \frac{h}{|t|^{\lambda_h/\lambda_t}} \right) + I_\pm(t) \right),$$ 

where the meanings of $\lambda_t, \lambda_h$ should be explained and the properties of the functions $I_\pm(t)$ given. What is the significance of the subscript label $\pm$?

Near to a critical point, for $h = 0$, the specific heat $C_V$ behaves like

$$C_V \sim \begin{cases} 
A_+ |t|^{-\alpha} & t > 0 \\
A_- |t|^{-\alpha} & t < 0 
\end{cases},$$

where $\alpha$ is the critical exponent. Explain why the amplitude ratio $A_+/A_-$ is expected to be universal.

Part III, Paper 49
The critical exponents $\beta, \gamma$ and $\nu$ are defined, for $h = 0$, by:

$$M \sim |t|^{\beta} \ (t < 0), \ \chi \sim |t|^{-\gamma}, \ \xi \sim |t|^{-\nu},$$

where $M$ is the magnetization and $\chi$ is the susceptibility. Establish the scaling relations $\alpha + 2\beta + \gamma = 2$ and $\alpha = 2 - D\nu$. 

Part III, Paper 49
The Gaussian model in $D$ dimensions for a real scalar field, $\phi(x)$, is defined by the Hamiltonian density

$$H(\phi(x)) = \frac{1}{2} \left[ \kappa^{-1} (\nabla \phi(x))^2 + m^2 \phi(x)^2 \right] - J(x) \phi(x),$$

where $\kappa$ and $m$ are coupling constants and $J(x)$ is a real external scalar field.

Find the Hamiltonian governing the Fourier transformed field $\tilde{\phi}(p)$, and show that the partition function for the system is given by

$$Z_0(J) = Z_G \exp \left( \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \tilde{\Delta}^{-1}(p) \tilde{J}(p) \tilde{J}(-p) \right),$$

where the meaning of $Z_G$ should be explained and the explicit form for $\tilde{\Delta}(p)$ given.

State what is meant by the two-point connected correlation function $\langle \tilde{\phi}(q) \tilde{\phi}(p) \rangle_c$ and, using the expression for $Z_0(J)$, show that

$$\langle \tilde{\phi}(q) \tilde{\phi}(p) \rangle_c = (2\pi)^D \delta^{(D)}(p + q) \tilde{G}_0(p),$$

where the explicit expression for $\tilde{G}_0(p)$ should be given in the limit $\tilde{J} \to 0$.

State how the inverse Fourier transform, $G_0(x)$, of $\tilde{G}_0(p)$ behaves as $|x| \to \infty$ and explain why the correlation length $\xi$ of the theory is given by $\xi \propto 1/m$.

Let the external source be a constant $J(x) = h$. Perform a suitable renormalization group transformation on the system that consists of a blocking step followed by a rescaling step, and determine how the coefficients $\kappa, m$ and $h$ transform.

The critical exponents $\alpha, \beta, \gamma$ and $\nu$ are defined by

$$C_V \sim |t|^{-\alpha}, \quad M \sim |t|^\beta (t < 0), \quad \chi \sim |t|^{-\gamma}, \quad \xi \sim |t|^{-\nu},$$

where the identification $m^2 \propto t$ is made for $t = (T - T_C)/T_C$ small and $T_C$ is the critical temperature. For $D < 4$ establish the relations

$$\alpha = (4 - D)/2, \quad \beta = (D - 2)/4, \quad \nu = 1/2, \quad \gamma = 1.$$

Verify that $\nu$ and $\gamma$ agree with their predicted values in Landau-Ginsberg theory.

For $D \geq 4$ explain why the prediction for $\alpha$ is $\alpha = 0$ which agrees with the prediction of Landau–Ginsberg theory.

END OF PAPER