UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 1:30 pm to 3:30 pm

PAPER 49

STATISTICAL FIELD THEORY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give an account of the Landau-Ginsberg (LG) theory of phase transitions in the context of a scalar field theory which should include a discussion of the following points:

(i) the idea of an order parameter;

(ii) the distinction between first-order and continuous phase transitions and how their defining properties are explained;

(iii) the idea of universality giving an example;

(iv) the idea of *critical exponents* and how they may be derived;

(v) the reason why a line of first order transitions must terminate in a critical point associated with a continuous phase transition;

(vi) the explanation of the features of a two-dimensional phase diagram containing a tricritical point in which you should identify a line of three-phase coexistence and give a reason for its occurence.

Near to a continuous phase transition at $T = T_C$, the critical exponents α, β, γ and δ are defined by

$$\begin{array}{ll} \text{Specific heat: } C_V \sim |t|^{-\alpha} \ (h=0), & \text{Magnetization: } M \sim |t|^{\beta} \quad (t<0, h=0), \\ \text{Susceptibility: } \chi \sim |t|^{-\gamma} \quad (h=0), & \text{Magnetization: } M \sim |h|^{1/\delta} \ (t=0), \end{array}$$

where $t = (T - T_C)/T_C$ and h is the applied magnetic field. Calculate α, β, γ and δ for a tricritical point and verify the scaling relations

$$\alpha + 2\beta + \gamma = 2, \quad \beta\delta = \beta + \gamma.$$

2

A spin model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin $\sigma_{\mathbf{n}}$ on the $\mathbf{n}$-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\boldsymbol{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\boldsymbol{u} = (u_1, u_2, \ldots)$. In particular, $\mathcal{H}$ contains the term $-h\sum_{\mathbf{n}} \sigma_{\mathbf{n}}$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\boldsymbol{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\boldsymbol{u},\sigma) - \beta N C) \;.$$

Define the two-point correlation function $G(\boldsymbol{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour as $|\boldsymbol{r}| \to \infty$. State how the susceptibility χ can be expressed in terms of $G(\boldsymbol{r})$.

Explain how the renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\boldsymbol{u}_p, C_p, N_p)$ which predicts the same large-scale properties for the system as does $\mathcal{Z}(\boldsymbol{u}, C, N)$. State how a and N rescale in terms of the RG scale factor b.

Derive the RG equation for the free energy $F(\boldsymbol{u}_p, C_p)$, and explain how it may be expressed in terms of a singular part, $f(\boldsymbol{u})$, which obeys the RG equation

$$f(\boldsymbol{u}_0) \;=\; b^{-pD} f(\boldsymbol{u}_p) \;+\; \sum_{j=0}^{p-1} b^{-jD}\, g(\boldsymbol{u}_j) \;, \qquad p > 0 \;.$$

What is the origin of the function $g(\boldsymbol{u})$ which determines the inhomogeneous part of the transformation?

Explain the idea of a fixed point, *relevant* and *irrelevant* operators, a critical surface and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived. In the case where there are two relevant couplings $t = (T - T_C)/T_C$ and h, and stating any assumptions you make, derive the scaling hypothesis for the free energy near to a critical point:

$$F(\boldsymbol{u}_0, C_0) \;=\; |t|^{D/\lambda_t}\left(f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) + C_\pm\right) + I_\pm(t) \;,$$

where the meanings of λ_t, λ_h should be explained and the properties of the functions $I_\pm(t)$ given. What is the significance of the subscript label $\pm$?

Near to a critical point, for $h = 0$, the specific heat C_V behaves like

$$C_V \sim \begin{cases} A_+|t|^{-\alpha} & t > 0 \\ A_-|t|^{-\alpha} & t < 0 \;, \end{cases}$$

where α is the critical exponent. Explain why the amplitude ratio A_+/A_- is expected to be universal.

The critical exponents β, γ and ν are defined, for $h = 0$, by:

$$M \sim |t|^{\beta} \ (t < 0), \quad \chi \sim |t|^{-\gamma}, \ \ \xi \sim |t|^{-\nu},$$

where M is the magnetization and χ is the susceptibility. Establish the scaling relations $\alpha + 2\beta + \gamma = 2$ and $\alpha = 2 - D\nu$.

3

The Gaussian model in D dimensions for a real scalar field, $\phi(\boldsymbol{x})$, is defined by the Hamiltonian density

$$\mathcal{H}(\phi(\boldsymbol{x})) = \frac{1}{2}\left[\kappa^{-1}(\nabla\phi(\boldsymbol{x}))^2 + m^2\phi(\boldsymbol{x})^2\right] - J(\boldsymbol{x})\phi(\boldsymbol{x}),$$

where κ and m are coupling constants and $J(\boldsymbol{x})$ is a real external scalar field.

Find the Hamiltonian governing the Fourier transformed field $\tilde{\phi}(\boldsymbol{p})$, and show that the partition function for the system is given by

$$\mathcal{Z}_0(\tilde{J}) = \mathcal{Z}_G \exp\left(\frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\tilde{J}(\boldsymbol{p})\tilde{\Delta}^{-1}(\boldsymbol{p})\tilde{J}(-\boldsymbol{p})\right) ,$$

where the meaning of $\mathcal{Z}_G$ should be explained and the explicit form for $\tilde{\Delta}(\boldsymbol{p})$ given.

State what is meant by the two-point connected correlation function $\langle\tilde{\phi}(\boldsymbol{q})\tilde{\phi}(\boldsymbol{p})\rangle_c$ and, using the expression for $\mathcal{Z}_0(\tilde{J})$, show that

$$\langle\tilde{\phi}(\boldsymbol{q})\tilde{\phi}(\boldsymbol{p})\rangle_c = (2\pi)^D\delta^{(D)}(\boldsymbol{p}+\boldsymbol{q})\tilde{G}_0(\boldsymbol{p}),$$

where the explicit expression for $\tilde{G}_0(\boldsymbol{p})$ should be given in the limit $\tilde{J} \to 0$.

State how the inverse Fourier transform, $G_0(\boldsymbol{x})$, of $\tilde{G}_0(\boldsymbol{p})$ behaves as $|\boldsymbol{x}| \to \infty$ and explain why the correlation length ξ of the theory is given by $\xi \propto 1/m$.

Let the external source be a constant $J(\boldsymbol{x}) = h$. Perform a suitable renormalization group transformation on the system that consists of a blocking step followed by a rescaling step, and determine how the coefficients κ, m and h transform.

The critical exponents α, β, γ and ν are defined by

$$C_V \sim |t|^{-\alpha}, \quad M \sim |t|^{\beta}\ (t<0), \quad \chi \sim |t|^{-\gamma}, \quad \xi \sim |t|^{-\nu},$$

where the identification $m^2 \propto t$ is made for $t = (T - T_C)/T_C$ small and T_C is the critical temperature. For $D < 4$ establish the relations

$$\alpha = (4-D)/2\ , \quad \beta = (D-2)/4\ , \quad \nu = 1/2\ , \quad \gamma = 1\ .$$

Verify that ν and γ agree with their predicted values in Landau-Ginsberg theory.

For $D \geqslant 4$ explain why the prediction for α is $\alpha = 0$ which agrees with the prediction of Landau–Ginsberg theory.

END OF PAPER