UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

PAPER 48

SYMMETRIES AND PARTICLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The strong interactions are invariant under isospin $SU(2)$ symmetry. Show how to construct the space of states on which a $(2I + 1)$-dimensional irreducible $SU(2)$ representation acts, where I is an integer or half-integer. You may assume the generators of the $SU(2)$ Lie algebra, J_i $(i = 1, 2, 3)$, satisfy $[J_i, J_j] = i\epsilon_{ijk}\, J_k$.

Show that the normalised states may be expressed in the form

$$|I, m\rangle = \sqrt{\frac{(I+m)!}{(2I)!(I-m)!}}\,(J_-)^{I-m}\,|I, I\rangle\,,$$

where $|I, I\rangle$ is a highest weight state and $J_- = J_1 - iJ_2$.

Consider a rotation, $R(\theta, \mathbf{n})$, through an angle θ around an axis defined by the unit vector $\mathbf{n}$ in isospin space. Define (but you need not derive) the unitary operator $U[R]$, expressed in terms of the generators J_i, that represents this rotation on states $|I, m\rangle$.

Define matrices with elements $D^{(I)}_{m'm}(\theta, \mathbf{n})$ in terms of $U[R]$ that furnish a $(2I+1)$-dimensional irreducible representation of $R(\theta\,, \mathbf{n})$. Demonstrate that your definition satisfies the necessary criteria for a representation.

Show that the matrix representation of J_2 in the $I = 1$ case is given by a 3×3 matrix, $J_2^{(1)}$, that satisfies $(J_2^{(1)})^3 = J_2^{(1)}$. Hence, or otherwise, show that a rotation $\theta = \pi$ around the $\mathbf{n} = (0, 1, 0)$ axis may be represented by

$$D^{(1)}_{m'm}(\mathbf{n}, \pi) = -(-1)^m\,\delta_{m',-m}\,,$$

and hence show that $e^{-i\pi J_2}\,|1, m\rangle = -(-1)^m\,|1, -m\rangle$.

Deduce how $e^{-i\pi J_2}$ acts on the three pi meson states, $|\pi^0\rangle$, $|\pi^+\rangle$ and $|\pi^-\rangle$, that form an $I = 1$ multiplet.

The 'charge conjugation' operator $\mathcal{C}$ that relates particle states to antiparticle states satisfies the relations

$$\mathcal{C}\,J_3\,\mathcal{C}^{-1} = -J_3\,, \qquad \mathcal{C}\,J_1\,\mathcal{C}^{-1} = -J_1\,, \qquad \mathcal{C}\,J_2\,\mathcal{C}^{-1} = J_2\,.$$

Show that $\mathcal{C}\,J_i\,\mathcal{C}^{-1}$ satisfies the $SU(2)$ algebra. Assuming that the $m = 0$ state of an isospin multiplet satisfies $\mathcal{C}|\pi^0\rangle = |\pi^0\rangle$, show that $\mathcal{C}|\pi^\pm\rangle = -|\pi^\mp\rangle$.

Show that the 'G-parity' operator $G = \mathcal{C}\,e^{-i\pi J_2}$ commutes with all the elements J_i and is therefore independent of the m quantum number. Determine the value of G for the pi meson states.

2

The group $SU(n)$ consists of $n \times n$ unitary complex matrices, $A^{\alpha}{}_{\beta}$, where $\alpha, \beta = 1, \ldots, n$. How does a $SU(n)$ (j,k)-tensor, $T^{\alpha_1 \ldots \alpha_j}_{\beta_1 \ldots \beta_k}$, transform?

Show that the complex conjugate of a (j,k) tensor is a (k,j) tensor.

Demonstrate that the tensors δ^{α}_{β}, $\epsilon^{\alpha_1 \ldots \alpha_n}$ and $\epsilon_{\beta_1 \ldots \beta_n}$ are invariant under $SU(n)$.

Show that the space of (j,k)-tensors contains invariant subspaces apart from the cases $(1,0)$, $(0,1)$ and $(0,0)$.

Show that the dimension of the $SU(n)$ representation formed by $(j,0)$-tensors that are symmetric on their upper indices is

$$\frac{(n+j-1)!}{j!(n-1)!}\,.$$

Determine the dimension of the space of tensors with j *anti*-symmetrised upper indices, $T^{[\alpha_1 \ldots \alpha_j]}$.

Explain why there is a *real* 6-dimensional representation of $SU(4)$.

Briefly explain why irreducible representations of symmetry groups are relevant for classifying elementary particles.

Discuss which irreducible representations of $SU(3)$ are present in the spectrum of low lying baryon states in QCD in the approximation that keeps only the light quarks (u, d, s). Ignore orbital angular momentum, but not quark spin.

3

Define the *adjoint representation* of a Lie algebra, $\mathfrak{g}$, of a group G of dimension d, and show that it satisfies the properties necessary for it to be a representation of $\mathfrak{g}$.

Consider a basis for a representation of $\mathfrak{g}$, T_a $(a = 1, \ldots, d)$. Let $e^{-X}\, Y e^{X} = (e^{X^{ad}})_{ab}\, Y_b\, T_a$, where $X = X_a\, T_a$ and $Y = Y_a\, T_a$ are elements of $\mathfrak{g}$ and $X^{ad} = X_a\, T_a^{ad}$ is the matrix corresponding to X in the adjoint representation. Show that the matrices $e^{X^{ad}}$ satisfy the conditions to form a representation of G.

Show that there is a basis in which the generators of $\mathfrak{g}$ may be identified with the structure constants of the Lie algebra.

Define the Killing form for a Lie algebra. Show that it is degenerate (has vanishing determinant) for a non-semi-simple algebra. Explain, without giving a proof, why the Killing form of a compact semi-simple algebra is strictly negative. [You may assume that the irreducible representations of a compact Lie group are isomorphic to unitary representations.]

A Lie algebra has basis vectors $\{X, Y, H\}$, where $[X, Y] = 2H$, $[X, H] = 2Y$ and $[Y, H] = 2X$. Find the adjoint representation of these elements and determine the Killing form of the algebra in this basis. Is this the algebra of a compact semi-simple Lie group?

Now consider the algebra $[X, Y] = 2H$, $[X, H] = -2Y$ and $[Y, H] = 2X$. Comment on the Killing form in this case.

4

Show how a Lorentz transformation of Minkowski space coordinates, $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ ($\mu = 0, \ldots, 3$), can be expressed in the form $X \to X' = A(\Lambda)\, X\, A(\Lambda)^{\dagger}$, where $A(\Lambda)$ is a $SL(2, C)$ matrix and $X = \sigma_{\mu} x^{\mu}$ (with $\sigma_{\mu} = (I, \sigma_i)$).

Show that the transformation given by the $SL(2, C)$ matrix

$$A_B(\theta, \mathbf{n}) = \cosh \frac{1}{2}\theta\, I + \sinh \frac{1}{2}\theta\, \sigma_i n_i$$

describes a Lorentz boost of x^{μ} along an axis defined by the unit three-vector $\mathbf{n} = \{n_i\}$ with velocity $\mathbf{v} = \tanh\theta\, \mathbf{n}$.

Demonstrate that the result of two successive Lorentz boosts with velocities $\mathbf{v}$ and $\mathbf{v}'$ along two distinct axes, $\mathbf{n}$ and $\mathbf{n}'$, is not purely a Lorentz boost unless $\mathbf{n} = \pm\mathbf{n}'$.

The Lie algebra of the Lorentz group can be written

$$[M_{\mu\nu}\, , M_{\rho\sigma}] = i\, (\eta_{\nu\rho}\, M_{\mu\sigma} - \eta_{\mu\rho}\, M_{\nu\sigma} + \eta_{\mu\sigma}\, M_{\nu\rho} - \eta_{\nu\sigma}\, M_{\mu\rho})\ ,$$

where $M_{\mu\nu} = -M_{\nu\mu}$ and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. Show that this may be decomposed into the sum of two subalgebras for the groups $SU(2)_L$ and $SU(2)_R$.

Verify that the $SL(2, C)$ matrix

$$A = I + \frac{1}{4}\, \omega^{\mu\nu}\, \sigma_{\mu}\, \bar{\sigma}_{\nu} + O(\omega^2)\, , \qquad\qquad (\omega^{\mu\nu} = -\omega^{\nu\mu})$$

where $\bar{\sigma}_{\mu} = (I, -\sigma_i)$, corresponds to the infinitesimal transformation $x^{\mu} \to x^{\mu} + \omega^{\mu}{}_{\nu}\, x^{\nu} + O(\omega^2)$. Deduce that there are two inequivalent two-component complex representations of $SL(2, C)$, ψ_L and ψ_R, and determine how they transform under Lorentz transformations with parameters $\Lambda^{\mu}{}_{\nu} = (\exp\omega)^{\mu}{}_{\nu}$.

Show that the four-component spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ transforms as

$$\psi \to \exp\left(\tfrac{1}{4}\, \omega^{\mu\nu}\, \gamma_{\mu\nu}\right)\, \psi\, ,$$

under Lorentz transformations, where $\gamma_{\mu\nu} = \frac{1}{2}\, (\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu})$ and γ_{μ} are 4×4 Dirac gamma matrices.

Prove that

$$e^{\frac{1}{4}\, \omega^{\mu\nu}\, \gamma_{\mu\nu}}\, \gamma^{\rho}\, e^{-\frac{1}{4}\, \omega^{\mu\nu}\, \gamma_{\mu\nu}} = \Lambda^{\rho}{}_{\mu}\, \gamma^{\mu}\, ,$$

and hence show that under Lorentz transformations: (i) $\bar{\psi}\, \psi$ transforms as a scalar; (ii) $\bar{\psi}\, \gamma^{\mu}\, \psi$ transforms as a vector; (iii) $\bar{\psi}\, i[\gamma^{\mu}\, , \gamma^{\nu}]\, \psi$ transforms as a second rank tensor.

[The Pauli matrices satisfy $\sigma_i\, \sigma_j = \delta_{ij}\, I + i\epsilon_{ijk}\, \sigma_k$, where I is the 2×2 unit matrix and $i = 1, 2, 3$. You may also assume that the Dirac matrices can be represented by

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \bar{\sigma}_{\mu} & 0 \end{pmatrix}\, ,$$

In this representation $\bar{\psi} = \psi^{\dagger}\, \gamma^0$ and $\gamma^{\mu\dagger}\, \gamma^0 = \gamma^0\, \gamma^{\mu}$.]

END OF PAPER