UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2012 9:00 am to 11:00 am

PAPER 47

SOLITONS AND INSTANTONS

*Attempt no more than **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Throughout this paper we define *dimensionless* spacetime coordinates $x_0 = t$ and $x_1 = x$. The sine-Gordon field theory is defined to be the theory of a dimensionless real scalar field $\phi(x,t)$ with Lagrangian density,

$$\mathcal{L} \;=\; \frac{m^2}{\beta}\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \,+\, (\cos\phi \,-\, 1)\right]\,,$$

where m is a mass scale and β is a dimensionless coupling. With these conventions the sine-Gordon equation reads

$$\partial_\mu\partial^\mu\phi \,+\, \sin\phi \;=\; 0\,.$$

1

The Bäcklund Transformation $\rho(x,t)$ of the function $\kappa(x,t)$ is defined by the equations,

$$\begin{aligned}\frac{\partial}{\partial x_+}\,(\rho \,-\, \kappa) \;&=\; 2a\,\sin\left(\frac{\rho+\kappa}{2}\right)\,,\\ \frac{\partial}{\partial x_-}\,(\rho \,+\, \kappa) \;&=\; \frac{2}{a}\,\sin\left(\frac{\rho-\kappa}{2}\right)\,,\end{aligned}$$

where a is an arbitrary real number and $x_\pm$ are lightcone coordinates in two-dimensional space time: $x_\pm = (x \pm t)/2$. Show that if $\kappa(x,t)$ obeys the sine-Gordon equation then so does $\rho(x,t)$.

Show that the sine-Gordon kink solution arises as the Bäcklund transformation of the vacuum $\kappa(x,t) = 0$ and derive a relation between the kink velocity and the parameter a.

Explain briefly how the existence of the Bäcklund Transformation is related to the *integrability* of the sine-Gordon equation.

Taking $\kappa(x,t) = 4\tan^{-1} e^x$ and $a = 1$, integrate the equations to find $\rho(x,t)$ and give a physical interpretation of this solution of the sine-Gordon equation.

2

Consider the theory of a single dimensionless complex scalar field $\psi(x,t)$ with Lagrangian density,

$$\mathcal{L} \;=\; M^2 \left[\frac{\partial_\mu \psi \partial^\mu \psi^*}{1 - \lambda^2 |\psi|^2} \;-\; |\psi|^2 \right] ,$$

where M is a mass scale and λ is a dimensionless coupling.

This theory has a stable, time-dependent soliton solution,

$$\psi_{\rm cl}(x,t) \;=\; \frac{\cos(\alpha)}{\lambda} \, \frac{\exp\left(i \sin(\alpha) t\right)}{\cosh\left(\cos(\alpha) x\right)}$$

for $-\pi/2 < \alpha \leqslant \pi/2$. [You are not required to verify that this expression solves the equations of motion.] Calculate the mass of this soliton and its charge under the $U(1)$ global symmetry: $\psi \to \exp(i\delta)\psi$, $\psi^* \to \exp(-i\delta)\psi^*$ for real δ.

By considering the symmetries of the Lagrangian, write down a more general soliton solution depending on four real parameters (including α).

Apply the Bohr-Sommerfeld condition to find a semiclassical quantisation condition for this family of solutions and determine the corresponding spectrum of particle masses as a function of $U(1)$ charge and the parameters M and λ. For a given value of λ, how many stable particle states are there?

[In this question you may use without derivation the following definite integral:

$$\int_{-\infty}^{+\infty} \frac{1}{\cosh^2(X) - \cos^2(\alpha)} \, dX \;=\; \frac{2}{\cos(\alpha)\sin(\alpha)} \left({\rm Sign}[\alpha] \frac{\pi}{2} \;-\; \alpha \right) . \quad]$$

3

Consider the following solution of the sine-Gordon equation,

$$\phi_{K\bar{K}}(x,t) \;=\; 4\,\tan^{-1}\left[\frac{\sinh\left(\frac{vt}{\sqrt{1-v^2}}\right)}{v\cosh\left(\frac{x}{\sqrt{1-v^2}}\right)}\right].$$

Explain its interpretation in terms of the scattering of a kink (K) with an anti-kink ($\bar{K}$). Hence calculate the leading semi-classical approximation to the phase shift $\delta_{K\bar{K}}(\theta)$ for K-$\bar{K}$ scattering as a function of difference θ of the K and $\bar{K}$ rapidities.

For special values of the sine-Gordon coupling β defined by

$$\begin{aligned}\gamma &= \beta^2\left(1-\frac{\beta^2}{8\pi}\right)^{-1}\\ &= \frac{8\pi}{N},\end{aligned}$$

where N is a positive integer, the exact transmission amplitude for K-$\bar{K}$ scattering is given as

$$S_T(\theta) \;=\; \exp\left(i\delta_{K\bar{K}}(\theta)\right) \;=\; e^{i\pi N}\prod_{k=1}^{N-1}\frac{e^{\theta-i\left(\frac{\pi k}{N}\right)}+1}{e^{\theta}+e^{-i\left(\frac{\pi k}{N}\right)}}.$$

The corresponding reflection amplitude vanishes: $S_R(\theta)\equiv 0$. From this result deduce the spectrum of K-$\bar{K}$ boundstates for each value of N, explaining your reasoning.

Show that the exact expression for $S_T(\theta)$ agrees with the result of your semiclassical computation in the limit $N\to\infty$. [*Hint: To demonstrate this agreement you may find it useful to consider the integral of the complex function* $\log\cos(z/2)$ *around a rectangular contour with corners at the points* $z=\pm i\theta$, $\pi\pm i\theta$.]

4

Write an essay on instanton effects in quantum mechanics. Your account should include a semiclassical analysis of tunneling in the double-well potential.

END OF PAPER