UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2012 9:00 am to 12:00 pm

PAPER 46

QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Lagrangian density for a complex scalar field $\phi(x)$ is

$$\mathcal{L} = \partial_\mu \phi^\dagger(x)\, \partial^\mu \phi(x) - m^2 \phi^\dagger(x)\, \phi(x)\,.$$

Obtain the fields $\pi(x)$, $\pi^\dagger(x)$ conjugate to $\phi(x)$, $\phi^\dagger(x)$. Write down the equal-time canonical commutation relations for these fields in the Heisenberg picture. Construct the Hamiltonian for the system. Why can $\phi(x)$ be expressed in the form

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E} \big(a(p) e^{-ip.x} - b^\dagger(p) e^{ip.x} \big)\,,$$

where $E = \sqrt{\mathbf{p}^2 + m^2}$, $p = (E, \mathbf{p})$ and the operators a, $a^\dagger$, b, $b^\dagger$ satisfy

$$\Big[a(p), a^\dagger(p') \Big] = (2\pi)^3\, 2E\, \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$
$$\Big[b(p), b^\dagger(p') \Big] = (2\pi)^3\, 2E\, \delta^{(3)}(\mathbf{p} - \mathbf{p}')\,,$$

all other commutators being zero?

The ground state of the system satisfies

$$a(p) \mid 0 \rangle = b(p) \mid 0 \rangle = 0\,.$$

Show that the Hamiltonian can be written as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2} \Big(a^\dagger(p)\, a(p) + b^\dagger(p)\, b(p) \Big)\,,$$

Explain the particle interpretation of the theory.

Verify that the current

$$j^\mu(x) = i \Big(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger) \phi \Big)$$

satisfies

$$\partial_\mu j^\mu = 0\,.$$

Show that the associated charge can be written as

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E} \Big(a^\dagger(p)\, a(p) - b^\dagger(p)\, b(p) \Big)\,.$$

Verify that

$$Q a^\dagger(p) = a^\dagger(p)(Q+1) \qquad Q b^\dagger(p) = b^\dagger(p)(Q-1)\,.$$

2

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0\,,$$

where the gamma matrices are given in the chiral representation by

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,,$$

where σ^i are the Pauli matrices and I_2 denotes the 2×2 unit matrix. Show that these matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_4\,,$$

where $g^{\mu\nu}$ is the Minkowski metric. Show that each component of the spinor $\psi(x)$ satisfies the Klein–Gordon equation.

Define the Lorentz group. A Lorentz transformation may be written as

$$\Lambda = \exp \frac{1}{2} \left(\Omega_{\rho\sigma} M^{\rho\sigma}\right)\,,$$

where $\Omega_{\rho\sigma}$ are parameters and $M^{\rho\sigma}$ is a set of constant 4×4 matrices. How many independent parameters are there? What are the physical meanings of these parameters?

The matrices may be chosen so that they satisfy the commutation relations

$$[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau} M^{\rho\nu} - g^{\rho\tau} M^{\sigma\nu} + g^{\rho\nu} M^{\sigma\tau} - g^{\sigma\nu} M^{\rho\tau}\,.$$

Show how one may use the Clifford algebra to construct a representation of the commutation relations. Use this to describe the action of the Lorentz transformation on the Dirac spinor ψ.

Explain why this representation of the Lorentz group cannot be unitary. Hence show that $\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$ is a Lorentz scalar. Is the interaction term $j_{5\mu} j_5^\mu$, where $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$, compatible with Lorentz invariance?

3

State Noether's theorem for a Lagrangian field theory. Give as an example a theory with a Lorentz four vector, $A_\mu(x)$.

Consider electromagnetism with gauge potential $A_\mu(x)$. What is the gauge transformation? Define the field tensor $F_{\mu\nu}$ and show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

is gauge invariant. Explain why there is a minus sign.

Show that $A_\mu \to A_\mu + \alpha\, \partial_0 A_\mu$, with α infinitesimal, is an infinitesimal symmetry of $\mathcal{L}$. Use Noether's theorem to find the conserved energy of the theory. An alternative symmetry is $A_\mu \to A_\mu + \alpha\,(\partial_0 A_\mu - \partial_\mu A_0)$. Use this to find a gauge invariant energy density. Comment on your result.

4

Consider the process

$$\mu^+ + \mu^- \to \gamma + \gamma\,.$$

Draw the leading order Feynman diagrams that contribute to this process. Write down the necessary Feynman rules. Use these to write down the corresponding scattering amplitudes.

Is there a constraint on the polarisation vector of the photon? Show that the scattering amplitude is unaffected if ϵ^μ is replaced by $(\epsilon + \alpha k)^\mu$, where α is a real constant. What is the physical interpretation of this result?

Now consider muon-photon scattering, $\mu^- + \gamma \to \mu^- + \gamma$. Write down the leading order diagrams and scattering amplitudes for this process. How does this differ from the first process?

END OF PAPER