UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 9:00 am to 11:00 am

PAPER 45

SUPERSYMMETRY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Please use the following conventions

$$\epsilon^{12} = -\epsilon_{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = +1,$$

$$(\theta\theta) \equiv \theta^{\alpha}\theta_{\alpha}, \qquad (\bar{\theta}\bar{\theta}) \equiv \theta_{\dot{\alpha}}\theta^{\dot{\alpha}}.$$

The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy

$$[\sigma_i,\ \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

If $M^{\mu\nu}$ are generators of the Lorentz group, the angular momentum operator is $J_k = \epsilon_{ijk}M^{ij}/2$, where $\{i,j,k\} \in \{1,2,3\}$. Defining $J_{\pm} = J_1 \pm iJ_2$, we find $[J_3, J_{\pm}] = \pm J_{\pm}$ and $[J^2, J_{\pm}] = 0$. Under a Lorentz transformation, a right-handed spinor $\bar{\chi}$ transforms as $\bar{\chi} \to \exp(-\frac{i}{2}w_{\mu\nu}\bar{\sigma}^{\mu\nu})\bar{\chi}$, where

$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})$$

and $\sigma^{\mu} = (1,\ \underline{\sigma})$ and $(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^{\mu})_{\beta\dot{\beta}}$, or $\bar{\sigma}^{\mu} = (1, -\underline{\sigma})$.

1

Assume that, under a Lorentz transformation,

$$\bar{Q}^{\dot{\alpha}} \rightarrow e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \quad \bar{Q}^{\dot{\alpha}} e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

when $\bar{Q}^{\dot{\alpha}}$ is viewed as a field operator.

Derive the canonical equations of $N = 1$ global supersymmetry algebra with the following left-hand sides:
(a) $[M^{\mu\nu}, \ \bar{Q}^{\dot{\alpha}}]$,
(b) $\{Q_\alpha, \ \bar{Q}_{\dot{\beta}}\}$.
Thus, derive the following relations, determining the constant numerical coefficients $a_{1,\ldots,9}$ in the process.

$$[J_i, \ \bar{Q}^{\dot{\alpha}}] = a_1(\sigma_i \bar{Q})^{\dot{\alpha}},$$

$$[J^2, \bar{Q}^{\dot{1}}] = a_2\bar{Q}^{\dot{1}} + \bar{Q}^{\dot{2}}(a_3J_1 + a_4J_2) + a_5\bar{Q}^{\dot{1}}J_3,$$

$$[J^2, \bar{Q}^{\dot{2}}] = a_6\bar{Q}^{\dot{2}} + \bar{Q}^{\dot{1}}(a_7J_1 + a_8J_2) + a_9\bar{Q}^{\dot{2}}J_3.$$

Consider a massive supermultiplet of mass m with elements labelled $|j, \ j_3\rangle$. Starting from a vacuum state $|\Omega\rangle = |0, \ 0\rangle$, use the above commutators and known facts about $J_\pm|j, \ j_3\rangle = (J_1 \pm iJ_2)|j, \ j_3\rangle$ to explicitly calculate the quantum numbers of $N\bar{Q}^{\dot{1}}|\Omega\rangle$. What is the value of the normalisation factor N?

Calculate the commutators $[J^2, \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}]$ and $[J_3, \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}]$, and then use them to determine the quantum numbers of the state $N^2\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$.

Using the above information, summarise all independent states in the supermultiplet, giving reasons for why there are no more.

2

Consider the $N = 1$ renormalisable globally supersymmetric model with superpotential $W(\Phi)$, where Φ is a chiral superfield:

$$W = \alpha + \kappa\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3!}\Phi^3.$$

(a) Provide arguments for why Φ may be shifted by a real number without changing any physical predictions of the theory. Thus, shift Φ to remove the linear term.
(b) Calculate the scalar potential of the model $V(\varphi)$, and sketch it against real values of φ.
(c) State whether the model breaks supersymmetry or not, along with your reasoning.
(d) Calculate the possible values of the vaccum expectation value $\langle\varphi\rangle$.
Suppose that φ is in one such non-trivial minimum $\langle\varphi\rangle \neq 0$. Working with the shifted field $\varphi = \tilde{\varphi} + \langle\varphi\rangle$,
(e) Calculate the scalar potential for $\tilde{\varphi}$.
(f) Thus, write down the mass of the complex scalar in terms of α, κ, m and g.
(g) By writing the rest of the potential, find the mass of the fermion in terms of α, κ, m and g.
(h) Is the ratio of the scalar mass to the fermion mass compatible with your answer to (c)?

3

Given J_i and K_i as the generators of angular rotations and Lorentz boosts, the Lorentz algebra

$$[K_i,\ K_j] = -i\epsilon_{ijk}J_k, \qquad [J_i,\ K_j] = i\epsilon_{ijk}K_k, \qquad [J_i,\ J_j] = i\epsilon_{ijk}J_k$$

holds. We may construct the new generators $A_k = \frac{1}{2}(J_k + iK_k)$ and $B_k = \frac{1}{2}(J_k - iK_k)$. Find the commutation relations

$$[A_i,\ B_j], \qquad [A_i,\ A_j] \qquad \text{and} \qquad [B_i,\ B_j],$$

in terms of A_i and B_i, and thus state explicitly which group they generate. What is the spin operator in terms of A_i and B_i?

The right-handed spinor representation of the Lorentz algebra is $\bar{\sigma}^{\mu\nu}$. Find J_i and K_i of the right-handed spinor representation in terms of the Pauli matrices σ_i.

By considering $\tilde{x} = x_\mu \sigma^\mu$, show that given $N \in SL(2, \mathbb{C})$, the expression

$$\Lambda^\mu_\nu(N) = \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^\mu N \sigma_\nu N^\dagger]$$

provides an explicit map from $SL(2, \mathbb{C})$ to $SO(1, 3)$.

Find C and D which satisfy, for left-handed Weyl spinors ψ_α and χ_α:
(a) $(\chi\psi) = C(\psi\chi)$
(b) $(\chi\psi)(\chi\psi) = D(\psi\psi)(\chi\chi)$.

END OF PAPER