SUPERSYMMETRY

Attempt no more than TWO questions.

There are THREE questions in total.

The questions carry equal weight.
Please use the following conventions
\[ \epsilon_{12} = -\epsilon_{12} = \epsilon_{12} = -\epsilon_{12} = +1, \]
\[ (\theta \bar{\theta}) \equiv \bar{\theta}^\alpha \theta_\alpha, \quad (\bar{\theta} \theta) \equiv \theta_\alpha \bar{\theta}^\alpha. \]

The Pauli matrices are:
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

They satisfy
\[ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k \]

If \( M^{\mu\nu} \) are generators of the Lorentz group, the angular momentum operator is
\[ J_k = \epsilon_{ijk} M^{ij}/2, \]
where \( \{i, j, k\} \in \{1, 2, 3\} \). Defining \( J_\pm = J_1 \pm iJ_2 \), we find \[ [J_3, J_\pm] = \pm J_\pm \]
and \[ [J^2, J_\pm] = 0. \]
Under a Lorentz transformation, a right-handed spinor \( \bar{\chi} \) transforms as
\[ \bar{\chi} \to \exp(-\frac{i}{2}w_{\mu\nu}\bar{\sigma}^{\mu\nu})\bar{\chi}, \]
where
\[ \bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \]
and \( \sigma^\mu = (1, \sigma) \) and \( (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}}, \) or \( \bar{\sigma}^\mu = (1, -\sigma) \).
Assume that, under a Lorentz transformation,
\[
\bar{Q}^\dot{\alpha} \rightarrow e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\bar{Q}^\dot{\alpha} e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}
\]
when \(\bar{Q}^\dot{\alpha}\) is viewed as a field operator.

Derive the canonical equations of \(N = 1\) global supersymmetry algebra with the following left-hand sides:
(a) \([M^{\mu\nu}, \bar{Q}^\dot{\alpha}]\),
(b) \(\{Q_\alpha, \bar{Q}^\dot{\beta}\}\).
Thus, derive the following relations, determining the constant numerical coefficients \(a_1, \ldots, 9\) in the process.
\[
[J^i, \bar{Q}^\dot{\alpha}] = a_1(\sigma_i \bar{Q})^\dot{\alpha},
\]
\[
[J^2, \bar{Q}^\dot{1}] = a_2 \bar{Q}^\dot{1} + \bar{Q}^\dot{2}(a_3 J^1 + a_4 J^2) + a_5 \bar{Q}^\dot{1} J^3,
\]
\[
[J^2, \bar{Q}^\dot{2}] = a_6 \bar{Q}^\dot{2} + \bar{Q}^\dot{1}(a_7 J^1 + a_8 J^2) + a_9 \bar{Q}^\dot{2} J^3.
\]

Consider a massive supermultiplet of mass \(m\) with elements labelled \(|j, j_3]\). Starting from a vacuum state \(|\Omega]\) = \(|0, 0]\), use the above commutators and known facts about \(J_\pm|j, j_3]\) = \((J^1 \pm iJ^2)|j, j_3]\) to explicitly calculate the quantum numbers of \(N\bar{Q}^\dot{1}|\Omega]\).

What is the value of the normalisation factor \(N^2\)?

Calculate the commutators \([J^2, \bar{Q}^\dot{1}\bar{Q}^\dot{2}]\) and \([J^3, \bar{Q}^\dot{1}\bar{Q}^\dot{2}]\), and then use them to determine the quantum numbers of the state \(N^2\bar{Q}^\dot{1}\bar{Q}^\dot{2}|\Omega]\).

Using the above information, summarise all independent states in the supermultiplet, giving reasons for why there are no more.
Consider the $N = 1$ renormalisable globally supersymmetric model with superpotential $W(\Phi)$, where $\Phi$ is a chiral superfield:

$$W = \alpha + \kappa \Phi + \frac{m}{2} \Phi^2 + \frac{g}{3!} \Phi^3.$$

(a) Provide arguments for why $\Phi$ may be shifted by a real number without changing any physical predictions of the theory. Thus, shift $\Phi$ to remove the linear term.
(b) Calculate the scalar potential of the model $V(\varphi)$, and sketch it against real values of $\varphi$.
(c) State whether the model breaks supersymmetry or not, along with your reasoning.
(d) Calculate the possible values of the vacuum expectation value $\langle \varphi \rangle$.
Suppose that $\varphi$ is in one such non-trivial minimum $\langle \varphi \rangle \neq 0$. Working with the shifted field $\varphi = \tilde{\varphi} + \langle \varphi \rangle$,
(e) Calculate the scalar potential for $\tilde{\varphi}$.
(f) Thus, write down the mass of the complex scalar in terms of $\alpha$, $\kappa$, $m$ and $g$.
(g) By writing the rest of the potential, find the mass of the fermion in terms of $\alpha$, $\kappa$, $m$ and $g$.
(h) Is the ratio of the scalar mass to the fermion mass compatible with your answer to (c)?
Given $J_i$ and $K_i$ as the generators of angular rotations and Lorentz boosts, the Lorentz algebra

\[ [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [J_i, J_j] = i\epsilon_{ijk}J_k \]

holds. We may construct the new generators $A_k = \frac{1}{2}(J_k + iK_k)$ and $B_k = \frac{1}{2}(J_k - iK_k)$. Find the commutation relations

\[ [A_i, B_j], \quad [A_i, A_j] \quad \text{and} \quad [B_i, B_j], \]

in terms of $A_i$ and $B_i$, and thus state explicitly which group they generate. What is the spin operator in terms of $A_i$ and $B_i$?

The right-handed spinor representation of the Lorentz algebra is $\bar{\sigma}^{\mu\nu}$. Find $J_i$ and $K_i$ of the right-handed spinor representation in terms of the Pauli matrices $\sigma_i$.

By considering $\tilde{x} = x_\mu \sigma^\mu$, show that given $N \in SL(2, \mathbb{C})$, the expression

\[ \Lambda^\mu_\nu(N) = \frac{1}{2}\text{Tr}[\bar{\sigma}^\mu N \sigma_\nu N^\dagger] \]

provides an explicit map from $SL(2, \mathbb{C})$ to $SO(1, 3)$.

Find $C$ and $D$ which satisfy, for left-handed Weyl spinors $\psi_\alpha$ and $\chi_\alpha$:

(a) $\langle \chi \psi \rangle = C \langle \psi \chi \rangle$
(b) $\langle \chi \psi \rangle \langle \chi \psi \rangle = D \langle \psi \psi \rangle \langle \chi \chi \rangle$.

END OF PAPER