UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 9:00 am to 11:00 am

PAPER 44

OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{P}$ be the set of probability measures on the measurable space $(E, \mathcal{E})$. Let $U_0 : \mathcal{P} \to \mathbb{R}$ be a fixed affine function, and define a preference relation $\succ$ on $\mathcal{P}$ by the rule

$$\lambda \succ \mu \Leftrightarrow U_0(\lambda) > U_0(\mu).$$

(a) Show that if $\lambda \succ \mu$ then

$$p\lambda + (1-p)\nu \succ p\mu + (1-p)\nu$$

for any $0 < p < 1$ and any $\nu \in \mathcal{P}$.

(b) Show that if $\lambda \succ \mu \succ \nu$ then there exists a $0 < p < 1$ such that

$$\mu \sim p\lambda + (1-p)\nu.$$

(c) Suppose $V_0 : \mathcal{P} \to \mathbb{R}$ is affine and has the property that

$$\lambda \succ \mu \Leftrightarrow V_0(\lambda) > V_0(\mu).$$

Show that there are constants $a > 0$ and b such that

$$V_0(\lambda) = aU_0(\lambda) + b$$

for all $\lambda \in \mathcal{P}$.

(d) Suppose that E is a finite set. Show that there exists a function $U : E \to \mathbb{R}$ such that

$$U_0(\lambda) = \int_E U(x)\lambda(dx).$$

2

Let U be a utility function that is twice-differentiable, strictly increasing and strictly concave on the interval $(0, \infty)$ and such that the Inada conditions hold. Let the dual function $\hat{U}$ be defined by

$$\hat{U}(y) = \sup_{x>0}[U(x) - xy].$$

(a) Show that $\hat{U}$ is twice-differentiable, strictly decreasing and strictly convex. Show that if $U'(x) = y$ then $\hat{U}'(y) = -x$ and $U''(x)\hat{U}''(y) = -1$.

(b) Let $P = (P_t)_{t\in\{0,1\}}$ be a one-period market model defined on a finite probability space, and let Z be a state price density. Show that that for any portfolio H such that $H \cdot P_0 = x$, the inequality

$$\mathbb{E}[U(H \cdot P_1)] \leqslant \mathbb{E}[\hat{U}(yZ)] + yx$$

holds for all $y > 0$. Hence, conclude that H^* is optimal for the problem

$$\text{maximise } \mathbb{E}[U(H \cdot P_1)] \text{ subject to } H \cdot P_0 = x,$$

if $U'(H^* \cdot P_1) = y^* Z^*$ a.s. for some constant $y^* > 0$ and state price density Z^*.

(c) Consider a two asset market with prices $P = (B, S)$. Both assets have initial price $B_0 = S_0 = 5$. The first asset is riskless with $B_1 = 6$, but the second is risky with distribution $\mathbb{P}(S_1 = 7) = 1/2 = \mathbb{P}(S_1 = 4)$. Find the unique state price density.

(d) An investor in the market described in part (c) has a utility maximising portfolio $H^* = (+4, -3)$, i.e. long 4 shares of the riskless asset and short 3 shares of the risky one. Assuming that his utility function is of the CRRA form $U(x) = \frac{x^{1-R}}{1-R}$ where $R > 0, R \neq 1$, find the relative risk aversion coefficient R.

3

Consider a discrete time market with two assets. The first is cash with price $B_t = 1$ a.s. for all $t \geqslant 0$. The second is a risky asset with prices $(S_t)_{t \geqslant 0}$.

(a) Explain why a self-financing investor's wealth X_t evolves as

$$X_t = X_{t-1} + \pi_t \Delta S_t$$

where $\Delta S_t = S_t - S_{t-1}$ and π_t is the number of shares of the risky asset held during the period $(t-1, t]$.

(b) Suppose $(\Delta S_t)_{t \geqslant 1}$ is a sequence of independent $N(\mu, \sigma^2)$ random variables, generating the filtration. Fix a time horizon $T > 0$ and find the optimal trading strategy for the problem

$$\text{maximise } \mathbb{E}[U(X_T)] \text{ over } (\pi_t)_{1 \leqslant t \leqslant T} \text{ given } X_0$$

where $U(x) = -e^{-\gamma x}$ for a constant $\gamma > 0$.

(c) The investor is now offered the opportunity to enter into an income swap contract. If he agrees to enter into the contract, he would receive the random income stream $(Y_t)_{1 \leqslant t \leqslant T}$ but must pay a fixed payment of y each period, so that his wealth evolves as

$$X_t = X_{t-1} + \pi_t \Delta S_t + Y_t - y.$$

Assume that $(Y_t, \Delta S_t)_{1 \leqslant t \leqslant T}$ is a sequence of independent random vectors generating the filtration, where

$$\begin{pmatrix} Y_t \\ \Delta S_t \end{pmatrix} \sim N\left(\begin{pmatrix} a \\ \mu \end{pmatrix}, \begin{pmatrix} b^2 & \rho b \sigma \\ \rho b \sigma & \sigma^2 \end{pmatrix} \right).$$

Find the optimal trading strategy for the investor if he enters into the swap contract.

(d) Show that the investor would prefer not to enter into the swap contract if $y > a - \mu \rho b / \sigma - \gamma b^2 (1 - \rho^2)/2$.

4

Let $(P_t)_{t\geqslant 0}$ be a n-dimensional Itô process modelling the prices of n assets.

(a) Explain why a self-financing investor's time-t wealth X_t satisfies the two equations

$$X_t = H_t \cdot P_t$$
$$dX_t = H_t \cdot dP_t - c_t dt$$

where H_t is the investor's portfolio and c_t is his consumption rate at time t.

(b) Let $(Z_t)_{t\geqslant 0}$ be a state price density. Show that

$$d(Z_t X_t) = H_t \cdot d(Z_t P_t) - Z_t c_t dt$$

(c) Assume that the pair (H, c) are such that the investor's wealth is always non-negative. Using standard results from stochastic calculus, show that the stochastic integral

$$M_t = \int_0^t H_s \cdot d(Z_s P_s)$$

defines a supermartingale.

(d) Prove that

$$\mathbb{E}\left(\int_0^\infty Z_t c_t \, dt\right) \leqslant X_0.$$

(e) Use Hölder's inequality to show that

$$\mathbb{E}\left(\int_0^\infty e^{-bt} U(c_t) dt\right) \leqslant U(X_0)\left[\mathbb{E}\left(\int_0^\infty e^{-bt/R} Z_t^{1-1/R} dt\right)\right]^R$$

where $U(c) = \frac{c^{1-R}}{1-R}$ and $0 < R < 1$.

END OF PAPER