UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

PAPER 43

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In a (one-factor) interest-rate model, the riskless rate r_t of interest solves the stochastic differential equation

$$dr_t = \sigma\sqrt{r_t}\, dB_t + (a - br_t)\, dt,$$

where B is a standard Brownian motion in the pricing measure, and σ, a and b are positive constants. Derive an explicit expression for the time-0 price of a zero-coupon bond which matures at $T > 0$.

Briefly discuss the main features, good and bad, of this particular model.

2

What is a *futures contract?* Briefly explain how a futures contract operates.

Supposing that a futures contract with expiry T is written on an underlying asset whose spot price at time t is S_t, derive an expression for the price F_{tT} at time $t < T$ of this futures contract. Explain the meaning of the terms *contango* and *backwardation.*

Suppose that the d-dimensional process X satisfies the linear stochastic differential equation

$$dX_t = dB_t - AX_t\, dt$$

where B is a standard d-dimensional Brownian motion in the pricing measure, and A is a $d \times d$ matrix. Find the form of the solution X as explicitly as you can, and specify the distribution of X_t.

If the spot price of the underlying asset is expressed as

$$S_t = \exp(b \cdot X_t)$$

for some fixed vector $b \in \mathbb{R}^d$, find the form of the futures price F_{tT} in terms of X_t as explicitly as you can.

3

Assume that $E(X) = E(X') = 0$.

Suppose that X and X' are two square-integrable random variables with positive variance. Define the correlation $\rho(X, X')$ between them. Prove that $\rho(X, X') = 1$ if and only if $X = aX'$ for some $a > 0$. (We say that X and X' are *perfectly correlated* if $\rho(X, X') = 1$).

Suppose that S_t and S'_t are two stock price processes, solving the stochastic differential equations

$$\begin{aligned} dS_t &= S_t\big(\, \sigma_t\, dB_t + r_t\, dt \,\big) \\ dS'_t &= S'_t\big(\, \sigma'_t\, dB_t + r_t\, dt \,\big) \end{aligned}$$

in the pricing measure, where σ, σ' and r are adapted processes, and B is a *common* standard Brownian motion.

(i) Assuming that the processes σ and σ' are both *constant*, prove that for each $t > 0$ the random variables S_t and S'_t are perfectly correlated if and only if $\sigma = \sigma'$.

(ii) Suppose that $r = 0$, and that σ and σ' are bounded, but not assumed constant. If $S_0 = S'_0 > 0$, and S_1 and S'_1 are perfectly correlated, prove that $\int_0^1 (\sigma_t - \sigma'_t)^2\, dt = 0$ almost surely.

(iii) If the processes σ and σ' are *not* assumed constant, and are supposed to be different (in the sense that $P(\int_0^t (\sigma_s - \sigma'_s)^2\, ds = 0) < 1$), can it be that S_t and S'_t are perfectly correlated? Explain your answer.

4

Suppose that X is a one-dimensional Ornstein-Uhlenbeck process, solving the stochastic differential equation

$$dX_t = \sigma dB_t - \lambda X_t \, dt$$

where σ and λ are positive constants, and suppose that

$$f(x) = a + \tfrac{1}{2}x^2$$

where $a > 0$. Find a condition on $\alpha > 0$ which guarantees that the process

$$\zeta_t \equiv e^{-\alpha t} f(X_t)$$

is a supermartingale.

Assuming this condition is satisfied, explain how the process ζ can be used to determine a pricing system. In this pricing system, express the riskless rate explicitly in terms of X, and find the time-0 price of a zero-coupon bond maturing at $T > 0$.

Comment briefly on the suitability of this model as a model for interest rates.

5

Suppose that $U : \mathbb{R} \to (-\infty, 0)$ is strictly increasing, concave and continuously differentiable. Let X be the (random) gain from investing in an asset from time 0 to time 1. An investor may choose any number θ, positive or negative, of units of the asset to hold from time 0 to time 1. The distribution of X is known at time 0, but not its actual value, which will be revealed at time 1. Suppose that the investor's objective

$$F(\theta) = E\,U(\theta X) \qquad (\theta \in \mathbb{R})$$

is everywhere finite-valued, and assume that the distribution of X is not degenerate.

(i) Prove that F is concave.

(ii) Prove that F is continuously differentiable with derivative

$$F'(\theta) = E[\,XU'(\theta X)].$$

(iii) Prove that *either* there exists some positive integrable Z such that $E[XZ] = 0$ *or* one of X, $-X$ is almost surely non-negative.

Now suppose that the investor has to pay a proportional non-negative transaction cost ε, so that the investor's objective changes to

$$G(\theta) = E\,U(\theta X - \varepsilon|\theta|),$$

again assumed finite-valued.

(i) Prove that G is concave.

(ii) Prove that G is differentiable except possibly at zero, with derivative

$$G'(\theta) = \begin{cases} E[\,U'(\theta(X-\varepsilon))(X-\varepsilon)] & (\theta > 0) \\ E[\,U'(\theta(X+\varepsilon))(X+\varepsilon)] & (\theta < 0) \end{cases}$$

(iii) Prove that *either* there exists some positive integrable Z such that $E[XZ] \in [-\varepsilon, \varepsilon]$; *or* one of $X - \varepsilon$, $-X - \varepsilon$ is almost surely non-negative.

6

(i) Suppose that $X_t = B_t + ct$, where B is a standard Brownian motion, c a constant, and set $H_a \equiv \inf\{t : X_t = a\}$ for $a > 0$. State without proof the density of H_a when $c = 0$. Stating carefully any results to which you appeal, show that for general c the density of H_a is

$$P(H_a \in dt)/dt = \frac{ae^{-(a-ct)^2/2t}}{\sqrt{2\pi t^3}},$$

and hence verify that

$$P(\, H_a \leqslant T\,) = \bar{\Phi}\left(\frac{a-cT}{\sqrt{T}}\right) + e^{2ac}\,\Phi\left(\frac{-a-cT}{\sqrt{T}}\right).$$

where Φ is the cumulative distribution function of the $N(0,1)$ distribution, and $\bar{\Phi}(x) = 1 - \Phi(x)$.

(ii) In a Black-Scholes model, the price S_t at time t of a stock is given by

$$S_t = \exp(\sigma B_t + (r - \tfrac{1}{2}\sigma^2)t\,)$$

where $\sigma > 0$ and r are constants. A bank sells a derivative which will pay 1 at the time $\tau \equiv \inf\{t : S_t > e^{\sigma a}\}$ if that time happens before the expiry $T > 0$, otherwise it pays nothing. Calculate the time-0 price of this derivative. (Assume $a > 0$.)

END OF PAPER