UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 42

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove the Lagrangian sufficiency theorem.

(b) Explain carefully how to solve the optimization problem

$$\begin{array}{ll} \text{minimize} & 3x_1 - x_2 + 2x_3^2 \\ \text{subject to} & x_1^2 + x_2^2 + x_3 = 2 \end{array}$$

You need not actually find a value for the Lagrange multiplier.

(c) Consider the optimization problem

$$\text{minimize} \quad \max\{a_i^T x + b_i : 1 \leqslant i \leqslant n\}$$

for constants $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ and variable $x \in \mathbb{R}^n$. Reformulate this problem as a linear program and derive the dual.

2

Consider the linear program

$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & 2x_1 + x_2 \geqslant 6 \\ & 2x_1 - x_2 \leqslant 0 \\ & x_1 + 2x_2 \leqslant 8 \\ & x_1, x_2 \geqslant 0. \end{array}$$

(a) Solve this problem using the two-phase simplex method and a pivoting rule that selects the variable with minimum index. Explain carefully what you are doing and derive the optimal solution and the corresponding objective value from the final tableau. Is there a pivoting rule for which the optimal solution is found more quickly?

(b) Use Gomory's cutting plane method to show that the problem becomes infeasible when x_1 and x_2 are required to be integral. Justify any additional constraints you derive.

3

The metric TSP is the special case of the traveling salesman problem in which distances satisfy the triangle inequality. You may assume that distances are symmetric.

(a) Define the class of NP-complete problems, and discuss what NP-completeness of a problem means for the existence of a polynomial-time algorithm. Define the concept of an α-approximation algorithm.

(b) Show that the metric TSP is NP-hard. What does this mean for the complexity of a version of the TSP where nodes may be visited more than once? You may use that the problem of deciding whether a graph has a Hamiltonian cycle is NP-complete.

(c) Show that there exists a 3/2-approximation algorithm for the metric TSP. To this end, consider the underlying graph $G = (V, E)$ of an instance of the metric TSP, let T be the arcs of a minimum spanning tree of G, and $U \subseteq V$ the set of nodes that have odd degree in (V, T). Show that the graph $(U, E \cap (U \times U))$ has a perfect matching M with cost at most $1/2$ of the cost of a minimum TSP tour, and that the graph $(V, T \uplus M)$ has an Euler tour, i.e., a walk that visits every arc exactly once and returns to the initial node. Here, $T \uplus M$ is the disjoint union of T and M, so the graph may contain up to two arcs between any pair of nodes.

4

(a) Let Γ be a bimatrix game with actions $M = \{1, \ldots, m\}$ for the row player and $N = \{m+1, \ldots, m+n\}$ for the column player. Denote the sets of (mixed) strategies of the row and column player by X and Y, respectively. Action $i \in M$ of the row player is said to be weakly dominated with respect to $A \subseteq M \cup N$ if there exists a strategy of the row player supported on A that offers at least the same payoff as i for every strategy of the column player supported on A and a strictly greater payoff for some such strategy, i.e., if there exists $x \in X$ with $S(x) \subseteq A$ such that $p(x, y) \geqslant p(e_i^m, y)$ for all $y \in Y$ with $S(y) \subseteq A$, and $p(x, y) > p(e_i^m, y)$ for some $y \in Y$ with $S(y) \subseteq A$, where e_i^m is the ith unit vector in $\mathbb{R}^m$. Weak dominance for the column player is defined analogously.

Consider a sequence of actions $a_1, \ldots, a_k \in M \cup N$ such that for all $i = 1, \ldots, k$, a_i is weakly dominated with respect to $A \setminus \{a_j : 1 \leqslant j < i\}$. Show that Γ has an equilibrium (x, y) with $S(x) \subseteq M \setminus \{a_j : 1 \leqslant j \leqslant k\}$ and $S(y) \subseteq N \setminus \{a_j : 1 \leqslant j \leqslant k\}$.

(b) Consider the bimatrix game with payoff matrices

$$P = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}.$$

Is this game non-degenerate? Use the Lemke-Howson algorithm to find an equilibrium of this game, and the payoffs of the two players in this equilibrium. Start by dropping the label corresponding to the third pure strategy of the row player. Explain carefully what you are doing.

5

(a) Define the core, the nucleolus, and the Shapley value of a coalitional game and provide intuitive descriptions of these solution concepts.

(b) Consider a game with a set N of players and characteristic function v, and let the dual game be the game with the same set of players and characteristic function v' such that for all $S \subseteq N$, $v'(S) = v(N) - v(N \setminus S)$. Show that the Shapley value s_i of each player $i \in N$ is the same in the game and the dual game. To this end, show that

$$s_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \big(v(N \setminus S) - v(S)\big),$$

and that the Shapley value of player i in the dual game is equal to this expression.

(c) Compute the nucleolus and the Shapley value of the game with characteristic function

$$\begin{aligned} v(\{1\}) &= 4, & v(\{2\}) &= 3, & v(\{3\}) &= 2, \\ v(\{1,2\}) &= 10, & v(\{1,3\}) &= 11, & v(\{2,3\}) &= 5, \\ & & v(\{1,2,3\}) &= 12. & & \end{aligned}$$

Show that the core of the game is empty.

6

Consider a setting with a set $A = \{1, \ldots, m\}$ of alternatives and a set $N = \{1, \ldots, n\}$ of voters, where every voter $i \in N$ has a strict linear order $\succ_i \subseteq A \times A$ indicating its preferences over alternatives in A.

(a) State the Gibbard-Satterthwaite Theorem and prove it for the case $n = 2$.

Preferences are called single-peaked if there exists a strict linear order $> \subseteq A \times A$ with the following property: for every $i \in N$, there exists an alternative $a_i \in A$, the peak for i, such that for any pair of alternatives $x, y \in A$, $a_i \geqslant x > y$ or $y > x \geqslant a_i$ implies that $x \succ_i y$. Intuitively, single-peakedness means that the alternatives can be arranged on a line such that alternatives on the same side of the peak a_i of agent i become less preferred with increasing distance from a_i.

(b) Show that a Condorcet winner is guaranteed to exist when preferences are single-peaked, and that the social choice function that selects the Condorcet winner is strategyproof.

END OF PAPER