UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 9:00 am to 11:00 am

PAPER 40

ACTUARIAL STATISTICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Suppose that the number of policies in a portfolio at the beginning of an accounting period is a random variable N, where $\mathbb{P}(N = n) = (1-p)^{n-1}p$, $n = 1, 2, \ldots$, for some $0 < p < 1$. Suppose that the number of policies is unchanged during an accounting period, and let X_i be the amount claimed on policy i during the accounting period, $i = 1, 2, \ldots, N$. Assume that the X_i are independent identically distributed positive random variables, independent of N. Let S be the total amount claimed on the portfolio during an accounting period.

(a) Derive an expression for the moment generating function $M_S(t)$ of S in terms of p and the moment generating function $M_X(t)$. Write down $\mathbb{P}(S = 0)$.

(b) Find the probability density function of S when the claim amounts are exponentially distributed with mean μ.

(c) In terms of p and μ, find the probability density function of S when the claim amounts have a gamma distribution with mean μ and variance $\mu^2/2$. Verify that your answer is indeed a probability density function.

2

For each of *quota share* and *excess of loss* reinsurance, state how much is paid by the direct insurer and by the reinsurer on each claim.

In a fixed time period, the total claim amount S (without reinsurance) has a compound Poisson distribution, $S \sim \mathrm{CP}(\lambda, F_X)$, with Poisson parameter λ and claim sizes with distribution function F_X and probability density function f_X. With reinsurance, let S_I and S_R be the total claim amounts paid by the direct insurer and the reinsurer, respectively, in the time period.

(a) For quota share reinsurance, find the value of the retained proportion α which minimises $\mathrm{var}(S_I) + \mathrm{var}(S_R)$.

(b) For excess of loss reinsurance, consider the function g that maps the retention M onto $\mathrm{var}(S_I) + \mathrm{var}(S_R)$. Show that $g'(M) = 0$ for M satisfying

$$M\big(1 - F_X(M)\big) = \int_M^\infty (x - M) f_X(x)dx.$$

Find the value of M that minimises $g(M)$ when the claims are exponentially distributed with mean μ.

(c) When $S \sim \mathrm{CP}(\lambda, F_X)$, let S_I and S_R be the direct insurer's and reinsurer's total claim amounts respectively under a general reinsurance contract. Let S_I^* and S_R^* be the direct insurer's and reinsurer's total claim amounts respectively under a quota share reinsurance contract with retained proportion α^* such that $\mathrm{var}(S_I^*) = \mathrm{var}(S_I)$. By writing $S_R = S - S_I$, or otherwise, show that

$$\mathrm{var}(S_I^*) + \mathrm{var}(S_R^*) \leqslant \mathrm{var}(S_I) + \mathrm{var}(S_R).$$

3

In a classical risk process, premiums arrive at rate c per unit time, and claims arrive in a Poisson process of rate λ per unit time. The claim sizes have probability density function $f(x)$. Let $\varphi(u)$ be the probability of never being ruined when the initial capital is $u \geqslant 0$. Show that

$$\varphi'(u) = a_1\varphi(u) + a_2 \int_0^u \varphi(u-x)f(x)dx,$$

where a_1 and a_2 are constants that you should specify.

Supose that $c = 2.1\lambda$ and

$$f(x) = \frac{1}{4}e^{-x/2}\left(2e^{-x/2}+1\right), \quad x > 0.$$

Show that

$$\varphi'(u) = b_1\varphi(u) + b_2e^{-u}I_1(u) + b_3e^{-u/2}I_2(u),$$

where $I_1(u) = \int_0^u \varphi(z)e^z dz$, $I_2(u) = \int_0^u \varphi(z)e^{z/2}dz$, and b_i, $i = 1,2,3$, are constants that you should specify. Show that

$$\varphi'''(u) + c_1\varphi''(u) + c_2\varphi'(u) = 0,$$

where c_1 and c_2 are constants that you should specify. Using that $\varphi(0) = \rho/(1+\rho)$ where ρ is the relative safety loading, and that $\varphi(u) \to 1$ as $u \to \infty$, find $\varphi(u)$ and hence the probability of ruin $\psi(u)$.

4

Explain briefly what is meant by a *credibility estimate* and a *credibility factor*.

Suppose that $X_1, \ldots, X_n$ are the total amounts claimed on a particular risk in years 1 to n, where, given a risk parameter θ, the X_i are independent and identically distributed with probability density function

$$f(x \mid \theta) = \frac{p(x)e^{-\theta x}}{q(\theta)} \quad x > 0, \tag{1}$$

for some functions $p(x)$ and $q(\theta)$. Let $\mu(\theta) = \mathbb{E}(X_i \mid \theta)$. Show that $\mu(\theta) = \frac{-q'(\theta)}{q(\theta)}$.

Suppose that θ has prior density

$$\pi(\theta) = \frac{q(\theta)^{-k}e^{-\theta\mu k}}{c(\mu, k)}, \quad \theta_0 < \theta < \theta_1, \tag{2}$$

for some $k > 0$ and $\mu > 0$, where $-\infty \leqslant \theta_0 < \theta_1 \leqslant \infty$. Assume that $\lim_{\theta \to \theta_i} \pi(\theta) = 0$, $i = 1, 2$. Show that the prior expectation of $\mu(\theta)$ is μ.

Suppose that $\mathbf{X} = (X_1, \ldots, X_n)$ is observed to be $\mathbf{x} = (x_1, \ldots, x_n)$. Show that, given $\mathbf{x}$, the posterior distribution for θ belongs to the same family of distributions as the prior, and write down the updated parameters of the posterior distribution. Find the posterior expectation of $\mu(\theta)$, and show that it can be written in the form of a credibility estimate.

Now suppose that $f(x \mid \theta) = \theta^2 x e^{-\theta x}$, $x > 0$. Specify $p(x)$ and $q(\theta)$ such that this probability density function may be written in the form of (1). Find the corresponding prior density in (2). Show that the resulting credibility factor is equal to

$$Z = \frac{n}{n + \frac{a}{b}},$$

where a is the prior expectation of $\mathrm{var}(X \mid \theta)$ and b is the prior variance of $\mu(\theta)$ (assume $k > 1/2$).

END OF PAPER