UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

# PAPER 4

# COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

*The questions carry equal weight.*

*All rings are understood to be commutative, unless stated otherwise.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Let $R$ be a reduced ring such that $\mathrm{Spec}(R)$ is not empty and not connected (as a topological space). Show that $R$ is isomorphic to the product of two nonzero rings.

(b) Let $R$ be a normal domain. Let $G$ be a finite group which acts on $R$ by automorphisms. Show that the ring of invariants $R^G = \{f \in R : g(f) = f \text{ for all } g \in G\}$ is normal.

**2**

Let $f : A^1_{\mathbf{R}} \to A^1_{\mathbf{R}}$ be the morphism of affine schemes over the real numbers $\mathbf{R}$ defined by $x \mapsto x^4$. For each closed point $p$ in $A^1_{\mathbf{R}}$, compute the number of irreducible components of the closed subscheme $f^{-1}(p)$. How many isomorphism classes of affine schemes over $\mathbf{R}$ arise as $f^{-1}(p)$ for closed points $p$ in $A^1_{\mathbf{R}}$?

**3**

Show that any prime ideal in $\mathbf{C}[x, y, z]$ of codimension $r$ can be generated by $r$ elements if $r = 1$ or $r = 3$. [You may use results from the course.]

On the other hand, show that the kernel of the $\mathbf{C}$-algebra homomorphism $\mathbf{C}[x, y, z] \to \mathbf{C}[t]$ given by $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$ is a codimension-2 prime ideal in $\mathbf{C}[x, y, z]$ that cannot be generated by 2 elements.

**4**

(a) Show that every vector space $V$ over a field $k$ is free as a $k$-module. [*Give a complete proof, without quoting any results from linear algebra. Note that $V$ is not assumed to be finite-dimensional.*]

(b) Let $f$ be an irreducible polynomial in $k[x_1, \ldots, x_n]$, for a field $k$ and a positive integer $n$. Show that the hypersurface $X = \{f = 0\} \subset A^n_k$ admits a finite flat morphism to affine $(n-1)$-space over $k$.

**5**

(a) Let $K$ be a finitely generated $\mathbf{Z}$-algebra which is a field. Show that $K$ is a finite field.

(b) Let $R$ be a finitely generated $\mathbf{Z}$-algebra. Show that the Jacobson radical (the intersection of all maximal ideals) of $R$ is equal to the nilradical of $R$. [*You may use (a) and results from the course, but do not use other results.*]

## END OF PAPER