UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 1:30 pm to 3:30 pm

PAPER 35

TOPICS ON CONFORMAL INVARIANCE AND RANDOMNESS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question, $(K_t, t \geqslant 0)$ will denote a chordal SLE$_\kappa$ from the origin to infinity in the complex upper half-plane $\mathbf{H}$. The goal will be to exhibit a special feature of the SLE processes for $\kappa \geqslant 8$.

(a) If $(W_t = \sqrt{\kappa}\beta_t, t \geqslant 0)$ denotes the driving function of the SLE, recall the equation describing the evolution of $g_t(z)$ for $z \in \mathbf{H}$ (where here, as in the lectures, g_t denotes the properly normalized conformal map from $\mathbf{H} \setminus K_t$ onto $\mathbf{H}$). Why is it possible to extend this (and the definition of the swallowing time $T(z) = \inf\{t \geqslant 0 \ : \ z \in \overline{K_t}\}$ of a point $z \in \mathbf{H}$, when it exists) to the case where z is real?

(b) When x is a given positive real number, we define $X_t := g_t(x) - W_t$. Show that when $\kappa > 4$, then $T(x)$ is almost surely finite. [*Hint: you might wish to study the process* $(X_t^{(1-(4/\kappa))})$.]

In the sequel, we will assume that $\kappa > 4$. For each $0 < x < y$, we define

$$F(x, y) = P(T(x) < T(y)).$$

(c) (i) When $\lambda > 0$ is given, show that $(K_{\lambda t}, t \geqslant 0)$ and $(\sqrt{\lambda} K_t, t \geqslant 0)$ have the same law; deduce from this that there exists a function f such that $F(x, y) = f(y/(y - x))$.

(ii) Describe the evolution of $Z_t := (g_t(y) - W_t)/(g_t(y) - g_t(x))$. Show that $f(Z_t)$ (up to some stopping time) is a martingale.

(d) (i) Describe all the positive C^2 functions h on $(1, \infty)$ such that $\lim_{z \to 1} h(z) = 0$ and for each $z > 1$,

$$\frac{\kappa}{2} h''(z) + 2\left(\frac{1}{z} + \frac{1}{z-1}\right) h'(z) = 0.$$

(ii) Show that these functions h are not bounded when $\kappa \geqslant 8$.

(iii) If h is such a function, prove that $h(Z_t)$ up to $T(x)$ is a non-negative local martingale.

(iv) Conclude that when $\kappa \geqslant 8$, then almost surely, $T(x) < T(y)$ for all $x < y$, while for $\kappa \in [4, 8)$, the probability that $T(x) = T(y)$ is strictly positive.

(e) We suppose that there almost surely exists a continuous curve $(\gamma_t, t \geqslant 0)$ in $\overline{\mathbf{H}}$ such that for all t, the complement of K_t in $\mathbf{H}$ is identical to the unbounded connected complement of $\mathbf{H} \setminus \gamma[0, t]$. Prove that almost surely, $\gamma(t)$ visits every point on the real line.

2

We denote by $\mathbf{L}$ the set of simple ("self-avoiding") loops in the complex plane, where we identify any two such loops that can be obtained from each other via monotone reparametrization. Equivalently, we view a loop γ as a compact subset of the complex plane.

We endow this set of loops with the σ-field of events $\mathcal{F}$ that is generated by the set of loops of the type

$$E_x := \{\gamma \in \mathbf{L} \ : \ x \text{ is in the bounded connected component of } \mathbf{C} \setminus \gamma\}.$$

For each simply connected domain U, we denote by $\mathbf{L}_U$ the set of self-avoiding loops that stay in U. We will assume that $\mathbf{L}_U \in \mathcal{F}$, and we will also assume that $\mathcal{F}$ is generated by the family of all such sets $\mathbf{L}_U$.

When μ is a measure on $\mathbf{L}$, we denote by μ_U the restriction of μ to the set of loops $\mathbf{L}_U$.

(a) We say that a σ-finite measure μ on $\mathbf{L}$ satisfies the property CR if for any simply connected domain U_1 and any conformal transformation Φ from U_1 onto $U_2 = \Phi(U_1)$, the image measure of μ_{U_1} under Φ is exactly μ_{U_2}. In other words, for all $A \in \mathcal{F}$,

$$\mu(\{\gamma \in \mathbf{L}_{U_1} \ : \ \Phi(\gamma) \in A\}) = \mu(\mathbf{L}_{U_2} \cap A).$$

Argue very briefly and informally why the outer boundary of the scaling limits of critical percolation clusters could be used to define a non-trivial measure μ that satisfies the property CR.

In the sequel, we are going to assume that there indeed exists such a measure μ.

(b) We shall now assume that ν is a (σ-finite) measure supported on the set E_0 of self-avoiding loops **that surround the origin**, such that for any simply connected domain U_1 that contains the origin and any conformal transformation Φ from U_1 onto $U_2 = \Phi(U_1)$ with $\Phi(0) = 0$, the image measure of ν_{U_1} under Φ is exactly ν_{U_2}. In other words, for all $A \in \mathcal{F}$,

$$\nu(\{\gamma \in \mathbf{L}_{U_1} \ : \ \Phi(\gamma) \in A\}) = \nu(\mathbf{L}_{U_2} \cap A).$$

For each simply connected subset U of the unit disc $\mathbf{U}$ that contains the origin, we denote by Φ_U the conformal map from the unit disc onto U with $\Phi_U(0) = 0$ and such that $\Phi_U'(0)$ is a positive real. We then define

$$A^\nu(\Phi_U) = \nu(\mathbf{L}_\mathbf{U} \setminus \mathbf{L}_U).$$

(i) Justify briefly the fact that the knowledge of all $A^\nu(\Phi_U)$ is sufficient to recover the measure $\nu_\mathbf{U}$.

(ii) Prove that $A^\nu(\Phi_{U_1} \circ \Phi_{U_2}) = A^\nu(\Phi_{U_1}) + A^\nu(\Phi_{U_2})$ for any pair (U_1, U_2) of simply connected subsets of the unit disc that contains the origin.

In the sequel, we are going to assume that:

- *The previous results imply that A^ν is necessarily of the type $A^\nu(\Phi_U) = -\lambda \log(\Phi'_U(0))$ for some $\lambda \geqslant 0$.*
- *There exists a measure ρ on planar Brownian loops B starting and ending at the origin, and that stay in the unit disc, such that for any simply connected subset U of the unit disc that contains the origin*

$$\rho(\{B \ : \ B \not\subset U\}) := -\log \Phi'_U(0)$$

(this measure has been described in the lectures, there is no need to recall its definition here).

- *For ρ-almost all loop B, the outer boundary of B is a self-avoiding loop.*

(iii) Is it possible to relate directly $\nu_{\mathbf{U}}$ to ρ? How (no need to work out the value of the multiplicative constants)?

(c) (i) Relate the measure $\mu_{\mathbf{U}}$ to ρ (no need to work out the value of the multiplicative constants).

(ii) Let E be a measurable set of self-avoiding loops that surround both 0 and $1/4$, and that stay in $\mathbf{U}/2$. Let us denote by $\gamma(B)$ (respectively $\gamma(B+1/4)$) the outer boundary of B (resp. of $B + 1/4$). Using the translation-invariance of μ, prove that

$$\rho(\{B \ : \ \gamma(B) \in E\}) = \rho(\{B \ : \ \gamma(B + 1/4) \in E\}).$$

3

(a) Describe how to rigorously define the law of a complex-valued Brownian motion B starting at the origin, that is "conditioned" to hit the unit circle at 1.

(b) We define B to be the previously described conditioned Brownian motion, and we let τ be its hitting time of 1. When U is a simply connected subset of the unit disc $\mathbf{U}$ that contains the origin and such that $d(\mathbf{U} \setminus U, 1) > 0$, we define the conformal map Φ_U from U onto $\mathbf{U}$ that maps 0 onto itself and the boundary point 1 onto 1.

Express the quantity $P(B[0,\tau) \subset U)$ in terms of Φ_U (provide a proper proof!).

(c) Outline how it is possible to relate some properties of B to a particular radial SLE process.

4

Write an essay on how it is possible to couple a Gaussian Free Field in the upper half-plane (with Dirichlet boundary conditions) with a chordal SLE_4 process from 0 to infinity in the upper half-plane.

A possible outline of the Essay can be:

- The definition of the Gaussian Free Field and of SLE_4.
- The statement of the coupling result.
- An outline of its proof.

END OF PAPER