UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 9:00 am to 12:00 pm

PAPER 34

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For this question, assume that S and T are two finite stopping times with $S \leqslant T$ and that M is an L^2-bounded martingale with quadratic variation process $[M]$.

(a) What is a previsible process? Show, from first principles, that $(\omega, t) \mapsto \mathbf{1}_{(S(\omega),T(\omega)]}(t)$ is a previsible process.

(b) Let H be a simple process. Define the stochastic integral $H \cdot M$ and prove that

$$\mathbb{E}[(H \cdot M)^2_\infty] = \mathbb{E}[(H^2 \cdot [M])_\infty]$$

[You may assume that the Lebesgue-Stieltjes integral on the right hand side is well-defined and that $H \cdot M$ and $M^2 - [M]$ are martingales.]

(c) Compute $(1_{(S,T]} \cdot M)_t$.

(d) Let H be a continuous, bounded and adapted process. If $M^{(T)}_s = M_{s+T} - M_T$, $s \geqslant 0$ then for any $t > 0$,

$$\int_T^{t+T} H_s dM_s = \int_0^t H_{T+s} dM^{(T)}_s.$$

In particular, you have to justify why the right hand side makes sense as a stochastic integral.

2

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$.

(a) Show that a continuous local martingale of finite variation starting from 0 must necessarily be identically 0, almost surely.

(b) Let X be a continuous semimartingale. Say what is meant by the quadratic variation process $[X]$ of X.

Suppose that $\mathbb{Q}$ is another probability measure on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$ and that X is also a semimartingale under measure $\mathbb{Q}$. Show that the quadratic variation of X under $\mathbb{Q}$ is indistinguishable (w.r.t. $\mathbb{Q}$) from the quadratic variation of X under $\mathbb{P}$. You may use without proof any standard characterization of quadratic variation.

(c) Let X be continuous process and let A be a continuous, increasing process with $X_0 = A_0 = 0$, almost surely. Suppose that for every $\theta \in \mathbb{R}$, the process defined by

$$Z_t^{(\theta)} = \exp\left(\theta X_t - \frac{1}{2}\theta^2 A_t\right), t \geqslant 0$$

is a local martingale. Argue that X and A are both adapted w.r.t. the filtration $(\mathcal{F}_t)_{t \geqslant 0}$. Prove that X is a local martingale and show that $[X]_t = A_t$, almost surely.

3

Let $B = (B^1, B^2)$ be a Brownian motion in $\mathbb{R}^2$ with $B_0 = (1,0)$. For $r \in (0,1)$ and $R \in (1, \infty)$, set

$$S_r = \inf\{t \geqslant 0 : |B_t| = r\}, T_R = \inf\{t \geqslant 0 : |B_t| = R\}.$$

(a) Use Itô's formula to show that $\log |B_{S_r \wedge t}|$ is a local martingale for any $r \in (0,1)$. [Here $|B_t|^2 = ((B_t^1)^2 + (B_t^2)^2)$.]

(b) Hence show that $\mathbb{E}[\log |B_{S_r \wedge T_R}|] = 0$ for any $r \in (0,1)$ and any $R \in (1, \infty)$. Carefully state any standard results that you use.

(c) Set $r_k = \frac{1}{k^k}$. Deduce from (b) that $\lim_{k \to \infty} \mathbb{P}[S_{r_k} \leqslant T_k] = 0$. Conclude that almost surely, $|B_t| > 0$ for all $t \geqslant 0$.

(d) Set $M_t = \log |B_t|$, which, by part (c), is well-defined. Show that M is a local martingale, but M is not a martingale.

4

(a) Let M be a continuous local martingale with $M_0 = 0$. What is the exponential local martingale of $\mathcal{E}(M)$? If $[M]_\infty \leqslant C$ almost surely for some constant C, show that $\mathcal{E}(M)$ is uniformly integrable.

(b) State Girsanov's theorem.

(c) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space and let B be a one-dimensional Brownian motion under $\mathbb{P}$. Let μ, ν be continuous bounded adapted processes and let σ be another continuous adapted process such that $\sigma_t \geqslant \delta$ for all $t \geqslant 0$ for some $\delta > 0$. Define

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s, \quad t \geqslant 0.$$

Fix $T > 0$. Using Girsanov's theorem, or otherwise, find a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that

$$X_t = \int_0^t \nu_s ds + \int_0^t \sigma_s d\tilde{B}_s, \quad t \leqslant T,$$

where $\tilde{B}$ is a Brownian motion under $\mathbb{Q}$.

5

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space and let B be a one-dimensional $(\mathcal{F}_t)_{t \geqslant 0}$ - Brownian motion defined on this probability space. Let $\sigma, b_1, b_2 : \mathbb{R} \to \mathbb{R}$ satisfy

$$|\sigma(x) - \sigma(y)| \leqslant K|x-y|, \quad |b_i(x) - b_i(y)| \leqslant L|x-y|, i = 1,2 \text{ for some } K, L.$$

Let X and Y be pathwise unique solutions of the following two SDEs:

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b_1(X_s)ds$$
$$Y_t = y_0 + \int_0^t \sigma(Y_s)dB_s + \int_0^t b_2(Y_s)ds.$$

Assume that $b_1(x) \leqslant b_2(x)$ for all $x \in \mathbb{R}$ and $x_0 < y_0$.

(a) Define $Z_t = Y_t - X_t$, $z_0 = y_0 - x_0$ and $\tau = \inf\{t \geqslant 0 : Z_t \leqslant 0\}$. Fix $\epsilon > 0$ and take $f(x) = \frac{1}{x+\epsilon}$. Using Itô's formula, find the semimartingale decomposition of $f(Z_{t\wedge\tau})$ and show that

$$\mathbb{E}[f(Z_{t\wedge\tau})] \leqslant f(z_0) + (L + K^2) \int_0^t \mathbb{E}[f(Z_{s\wedge\tau})]ds.$$

(b) Using part (a), show that

$$\mathbb{E}[f(Z_{t\wedge\tau})] \leqslant f(z_0)e^{(L+K^2)t},$$

and hence conclude that $\mathbb{P}[X_t \leqslant Y_t \text{ for all } t \geqslant 0] = 1$.

6

(a) Let X be a continuous adapted process defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ with values in $\mathbb{R}^d$, and let $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}, b : \mathbb{R}^d \to \mathbb{R}^d$ be bounded measurable functions. Explain what you mean by the statement: X solves the martingale problem $\mathbf{M}(a, b)$ associated with a and b. When do we say that the martingale problem $\mathbf{M}(a, b)$ is well-posed?

(b) Let X be an adapted continuous process with values in $\mathbb{R}^d$. Suppose that

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a local martingale for every function $f \in C^2(\mathbb{R}^d)$, where

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x^i}.$$

Prove that X is a solution to the martingale problem $\mathbf{M}(a, b)$.

(c) Let $n \geqslant 1$. Let $(Z^n_k)_{k\geqslant 0}$ be the discrete time Markov chain on the state space $\{0, 1, 2, \ldots, n\}$ whose one step dynamics is described as below: At time k, put Z^n_k red balls and $n - Z^n_k$ blue balls in an urn.

Draw a ball from the urn uniformly at random, note its colour and put the ball back to the urn. Repeat n times.

Z^n_{k+1} is the number of red balls in the n draws done as above.

Assume that $Z^n_0 = \lfloor xn \rfloor$, where $x \in (0, 1)$. $X^n_t = Z^n_{\lfloor nt \rfloor}/n$. Show that

$$(X^n_t, t \in [0, 1]) \to (X_t, t \in [0, 1]),$$

weakly, where X is the solution of the stochastic differential equation

$$dX_t = \sqrt{X_t(1 - X_t)}dB_t, \quad X_0 = x,$$

where B is a one-dimensional Brownian motion. You may assume without proof that the above SDE has a unique solution.

State carefully the results that you are applying.

END OF PAPER