UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 1:30 pm to 4:30 pm

PAPER 33

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let (X_i) be i.i.d. random variables with $\mathbb{P}(X_1 = +1) = \mathbb{P}(X_1 = -1) = 1/2$. Define $S_0 = 0$ and for all $n \geqslant 1$ let $S_n = \sum_{i=1}^{n} X_i$.

(a) Let $T_1 = \min\{n \geqslant 0 : S_n = 1\}$. Explain why T_1 is a stopping time and calculate its expectation.

(b) Construct a martingale which converges almost everywhere but not in L^1.

[*Hint: Use the stopping time T_1 from (a) in the construction.*]

(c) Let $T = \min\{n \geqslant 2 : S_n = S_{n-2} + 2\}$ and $U = T - 2$. Are T and U stopping times? Justify your answers.

(d) For T as defined in (c), what is $\mathbb{E}[T]$?

[You may assume any form of the Optional Stopping Theorem provided you state it clearly.]

2

(a) State Cramer's Theorem on the large deviations of the sample mean of a sequence of i.i.d random variables.

(b) Let (Y_n) be a sequence of i.i.d. Exponential(1) random variables. Define $S_n = \sum_{i=1}^{n} Y_i$. Show that for all $a \geqslant 0$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant na)$$

exists and determine its value.

(c) Let (U_n) be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. Set $X_n = U_n^{-1/2}$ and $W_n = \sum_{k=1}^{n} X_k$. Show that for all $a \geqslant 0$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(W_n \geqslant na)$$

exists and determine its value.

3

(a) Give the definition of a Poisson random measure.

(b) Let Φ and Ψ be two independent Poisson random measures of intensities μ and ν respectively. Show that $\Phi + \Psi$ is also a Poisson random measure and find its intensity.

(c) Let (Z_i) be the arrival times of a Poisson process of intensity 1 on the interval $[0, \infty)$, so that the Poisson process $N([0,t])$ is the number of arrivals in $[0,t]$. Define $X = \sum_i Z_i^{-2}$. Let (W_i) be the arrival times of another Poisson process independent of the first one and define $Y = \sum_i W_i^{-2}$. Show that $X + Y$ has the same law as cX for some constant c and find the value of c.

(d) Let $\Pi = \{X_i\}$ be a Poisson random measure in $\mathbb{R}^d$ whose intensity is the standard Lebesgue measure on $\mathbb{R}^d$. Write $X_1, X_2, \ldots$ for the positions of the atoms of Π, listed in increasing order of their distance from 0. Let $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a deterministic continuous function and let $(\xi_i(t))$ be independent standard Brownian motions in d dimensions. We now consider the atoms moving with Brownian motion, with trajectories of the form $(X_i + \xi_i(t))_{t \geqslant 0}$. We define

$$T = \inf\{t \geqslant 0 : f(t) \in \cup_i \mathcal{B}(X_i + \xi_i(t), 1)\}\,.$$

Show that

$$\mathbb{P}(T > t) = \exp(-\mathbb{E}[\mathrm{vol}(W(t))])\,,$$

where $W(t) = \cup_{s \leqslant t} \mathcal{B}(\xi(s) + f(s), 1)$.

[You may assume the thinning property, but any other result needs to be proved.]

4

(a) State and prove Doob's maximal inequality.

Let (X_i) be $N(-1,1)$ i.i.d. random variables and let $S_n = X_1 + \ldots + X_n$ and $S_0 = 0.$. Write $Z = \max_{i \geqslant 0} S_i$.

(b) Show there is a unique $\lambda > 0$ so that $(e^{\lambda S_n})$ is a martingale.

(c) Show that $\mathbb{P}(Z > t) \leqslant e^{-\lambda t}$.

(d) Show that $\mathbb{E}[e^{bZ}]$ is finite for all $b < \lambda$.

5

(a) Let B be a Brownian motion in $d = 2$ dimensions and let $x \in \mathbb{R}^2$ such that $r < |x| < R$, where $0 < r < R$. For all $a \geqslant 0$ we define

$$T_a = \inf\{t \geqslant 0 : |B_t| = a\}\,.$$

Show that

$$\mathbb{P}_x(T_r < T_R) = \frac{\log R - \log|x|}{\log R - \log r}\,.$$

(b) Show that planar Brownian motion does not hit points. In particular, show that planar Brownian motion never returns to its starting point. Show further that, for any open set U containing the starting point, the set of times at which planar Brownian motion visits U is unbounded.

(c) Show that Brownian motion in 3 dimensions does not hit lines, i.e. if it is started outside a line, then a.s. it will never hit it.

6

(a) Let (B_s) be a standard Brownian motion in one dimension. Show that $t^\alpha \mathbb{P}(B_s \leqslant 1, \forall s \leqslant t)$ converges to a finite positive constant as $t \to \infty$ for some α and find the value of α and the constant.

(b) State Donsker's invariance principle.

(c) What is the limit of $\mathbb{P}(\max_{j \leqslant n} S_j \geqslant a\sqrt{n})$ as $n \to \infty$ for $a > 0$, where (S_j) is a simple random walk on $\mathbb{Z}$. Justify your answer.

END OF PAPER