UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 32

STOCHASTIC NETWORKS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State Little's Law, giving a brief description of the three terms appearing in its statement.

By considering the system comprising just the server at a stationary $M/M/1$ queue, use Little's Law to show that the probability the server is busy is $\lambda\mu$, where λ is the arrival rate at the queue, and μ is the mean service time of a customer.

Customers arrive according to a Poisson process with rate λ at a single server, but a restricted waiting room causes those who arrive when n customers are already present to be lost. Accepted customers have service times which are independent and identically distributed with mean μ and independent of the arrival process. (Service times are not necessarily exponentially distributed.) If P_j is the long-run proportion of time j customers are present, show that

$$1 - P_0 = \lambda\mu(1 - P_n)\,.$$

2

Derive Erlang's formula $E(\nu, C)$, for the proportion of calls lost at a resource of capacity C offered a load of ν. State clearly any assumptions you make in your derivation.

Define a loss network with fixed routing, and describe briefly how the equations

$$B_j = E\left(\sum_r A_{jr}\nu_r \prod_{i\in r-\{j\}} (1 - B_i), \quad C_j\right), \qquad j = 1, 2, \ldots, J \tag{1}$$

arise as a natural approximation for the link blocking probabilities in the network.

Establish the existence and uniqueness of a solution to the equations (1).

3

Briefly outline a mathematical model of the slotted infinite-population ALOHA random access scheme, obtaining the recurrence

$$N_{t+1} = N_t + Y_t - I[Z_t = 1]\,,$$

for N_t, the backlog of packets awaiting retransmission, where $Z_t = 0, 1$ or $*$ accordingly as 0,1 or more than 1 packets are transmitted in slot $(t, t+1)$, and Y_t is the number of arrivals in slot $(t-1, t)$. Let f be the probability a backlogged packet is retransmitted. Explain why N_t is a Markov chain and determine its transition probabilities.

Prove that for any positive arrival rate

$$P\{\exists J < \infty : Z_t = *, \quad \text{for all } t \geqslant J\} = 1\,.$$

Briefly outline at least one other random access scheme, where the retransmission probability of a packet is not constant.

4

Write an essay on the concept of an *effective bandwidth.* Your essay should include, but not be limited to, a derivation of Chernoff's bound, and an interpretation of the expression

$$\alpha(s) = \frac{1}{s} \log \mathbb{E}[e^{sX}]$$

when the parameter s is (i) small, (ii) large.

5

The dynamical system

$$\begin{aligned} \frac{d}{dt}\mu_j(t) &= \kappa_j \mu_j(t) \left(\sum_{r:j\in r} x_r(t) - q_j(\mu_j(t)) \right) && j \in J \\ x_r(t) &= \frac{w_r}{\sum_{k\in r} \mu_k(t)} && r \in R \end{aligned}$$

is proposed as a model for resource allocation in a network, where R is set of routes, J is a set of resources, and for $j \in J$, $\kappa_j > 0$, and $q_j(\cdot)$ is a continuous, strictly increasing function with $q_j(0) = 0$.

Provide a brief interpretation of this model, in terms of price signals generated by resources and acted upon by routes.

By considering the function

$$V(\mu) = \sum_{r\in R} w_r \log \left(\sum_{j\in r} \mu_j \right) - \sum_{j\in J} \int_0^{\mu_j} q_j(\eta) d\eta$$

or otherwise, show that all trajectories of the dynamical system converge toward a unique equilibrium point.

END OF PAPER