UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 1:30 pm to 3:30 pm

PAPER 30

PERCOLATION AND RELATED TOPICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $(x_n : n \geqslant 1)$ be a real sequence satisfying $x_{m+n} \leqslant x_m + x_n$ for all $m, n \geqslant 1$. Show that the limit $\lambda = \lim_{n\to\infty} x_n/n$ exists and satisfies $\lambda = \inf\{x_m/m : m \geqslant 1\}$.

(b) Let c_n be the number of self-avoiding walks on the square lattice $\mathbb{Z}^2$ starting at the origin and of length n. Show that the limit $\kappa = \lim_{n\to\infty} c_n^{1/n}$ exists and satisfies $2 \leqslant \kappa \leqslant 3$.

(c) Show that $\kappa > 2$. [You might find it useful to consider paths whose vertical steps are always upwards.]

2

(a) Let P be a product measure on the finite space $\Omega = \{0,1\}^E$. Let A_1, A_2 be increasing events with equal probability. Show that

$$P(A_1) \geqslant 1 - \sqrt{1 - P(A_1 \cup A_2)}\,.$$

[You may appeal without proof to any general theorem.]

(b) Consider bond percolation on the square lattice with edge-density p. Define the *percolation probability* $\theta(p)$. Assume that $p = \frac{1}{2}$ and show that $\theta(\frac{1}{2}) = 0$. You may use the result of Question 4, and you may wish to follow the preliminary steps below.

Suppose $\theta(\frac{1}{2}) > 0$, and write $B_n = [-n, n]^2$.

(i) Let $\epsilon > 0$. Show that there exists $N = N(\epsilon)$ such that, with probability greater than $1 - \epsilon$, the boundary of B_N intersects an infinite open cluster.

(ii) There exists N such that, with probability greater than $\frac{3}{4}$, each horizontal side of B_N intersects an infinite open cluster in the complement of B_N.

(c) Let $p \in (0,1)$. The edges of the square lattice are directed at random, independently of one another. Horizontal edges are directed eastwards with probability p and otherwise westwards, vertical edges are directed northwards with probability p and otherwise southwards. Let $\psi(p)$ be the probability that the origin is the first vertex of an infinite directed path. Show that $\psi(\frac{1}{2}) = 0$.

3

(a) Let μ_1, μ_2 be probability measures on the finite space $\Omega = \{0,1\}^E$. Explain what is meant by saying that μ_1 dominates μ_2 stochastically, written $\mu_1 \geqslant \mu_2$.

(b) State Holley's condition for the inequality $\mu_1 \geqslant \mu_2$.

(c) Let $G = (V, E)$ be a finite graph and let $\mu_{p,q}$ denote the random-cluster measure on G with parameters $p \in (0,1)$ and $q \geqslant 1$. Show that $\mu_{p,q}(A)$ is non-increasing in q, for any increasing event A.

(d) Let $p_{\mathrm{c}}(q)$ be the critical point of the random-cluster model on the square lattice. Show that $p_{\mathrm{c}}(q)$ is non-decreasing in $q \geqslant 1$.

4

Write a general account of the proof of the uniqueness of the infinite cluster in bond percolation on the hypercubic lattice $\mathbb{Z}^d$ with $d \geqslant 2$.

Your answer should include a clear statement of the theorem in question, together with accounts and explanations of the principal steps in the proof. The emphasis should be on explaining the proof rather than on giving all the details.

END OF PAPER