UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 9:00 am to 12:00 pm

PAPER 3

REPRESENTATION THEORY AND INVARIANT THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $A = \mathbb{C}S_n$ be the group algebra of the symmetric group. Define the terms *partition* λ of n, *Young tableau* t_λ of shape λ and *Young symmetrizer* $h(t_\lambda)$ of a given tableau. Define the *dictionary order* $\geqslant$ on the set of partitions of n.

If $\lambda \geqslant \mu$ are partitions of n, and t_λ and t_μ are Young tableaux of shape λ and μ, respectively, show that one of the following statements is true:

(A) There are distinct integers i and j which occur in the same row of t_λ and the same column of t_μ.

(B) $\lambda = \mu$ and $t_\mu = rc\, t_\lambda$ for some element r in the row stabiliser of t_λ and some element c in the column stabiliser of t_λ.

For fixed n, show that the set of left ideals of A of the form Ah_λ, where h_λ is a certain Young symmetrizer, and λ runs through the partitions of n, is a complete set of non-isomorphic irreducible A-modules (called the *Specht modules*).

[Standard facts about semisimple algebras may be assumed, as can basic properties of Young symmetrizers.]

2

In what follows below we will use the notation of question 1. Also, standard facts about semisimple algebras may be assumed, as can basic properties of Young symmetrizers.

Let $A = \mathbb{C}S_n$. Let λ be a partition of n. What is a *standard λ-tableau*? Define the usual order $\leqslant$ on the standard tableau of a given shape.

Show that, if $t_\lambda > t'_\lambda$ are standard tableaux then $h(t_\lambda)\, h(t'_\lambda) = 0$.

Prove that

$$A \cong \bigoplus Ah(t_\lambda)$$

where the (direct) sum is over all standard tableau of shape λ, as λ runs through the partitions of n.

Deduce a result about the dimension of the Specht module associated to λ.

3

Let V be a vector space of dimension m and let $n \in \mathbb{N}$. Define the usual actions of S_n and $\mathrm{GL}(V)$ on tensor space $T^n V = V^{\otimes n}$. Show that these two actions commute.

Define the *Schur algebra* $S_{\mathbb{C}}(m,n)$, which you may assume to be a semisimple $\mathbb{C}$-algebra.

State and prove Schur-Weyl duality. [You may assume basic results from algebraic geometry, and basic results from multilinear algebra.]

4

Let V be an m-dimensional vector space over $\mathbb{C}$.

If $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$ with weakly decreasing parts, define the $\mathbb{C}\mathrm{GL}(V)$-module $D_\lambda = D_{\lambda_1,\ldots,\lambda_m}(V)$. If you wish, you may assume these modules are irreducible rational $\mathbb{C}\mathrm{GL}(V)$-modules. Denote by ϕ_λ its character.

Let α be an S_n-conjugacy class with cycle type $n^{\alpha_n} \ldots 1^{\alpha_1}$ and let ξ be an endomorphism of V with eigenvalues $x_1, \ldots, x_m$. If s_i is the power sum, show that

$$s_1^{\alpha_1} \cdots s_n^{\alpha_n} = \sum_{\lambda \in \Lambda^+(n,m)} \chi^\lambda(\alpha) \phi_\lambda(\xi)$$

where summation is taken over the partitions λ of n into at most m parts, and where χ^λ is the character of the corresponding Specht module. [Results about semisimple modules and about characters of the symmetric group can be assumed.]

State Weyl's character formula for the general linear group and deduce it using the previous result.

Deduce the precise form of the m-tuple μ such that the dual of $D_{\lambda_1,\ldots,\lambda_m}(V)$ is isomorphic to D_μ.

5

Let W be a finite-dimensional vector space over $\mathbb{C}$ with coordinate ring $\mathbb{C}[W]$. Let G be a group. If W is a $\mathbb{C}G$-module, explain how $\mathbb{C}[W]$ inherits a $\mathbb{C}G$-structure. Define the *polynomial invariant ring* $\mathbb{C}[W]^G$. Define also the discriminant of a monic polynomial $f(t) \in \mathbb{C}[t]$.

Let A_n denote the alternating group of degree ($n \geqslant 2$). Quoting any facts you may need about symmetric polynomials, show that $\mathbb{C}[X_1, \ldots, X_n]^{A_n}$ is a quotient of a polynomial algebra in $n+1$ indeterminates.

Let $\rho : G \to \mathrm{GL}(W)$ be a representation of any finite group G. Suppose that there is no non-trivial homomorphism $G \to \mathbb{C}^\times$. Show directly that $\mathbb{C}[W]^G$ is a unique factorisation domain. By considering a representation of the cyclic group of order 2, or otherwise, show that a polynomial invariant ring can fail to be a unique factorisation domain, even in characteristic zero.

END OF PAPER