UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 29

PRIME NUMBERS

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Möbius inversion formula.

State the prime number theorem with classical error term.

Deduce that

$$\frac{1}{X}\Big|\sum_{n\leqslant X}\mu(n)\Big| \ll 1/\log X.$$

2

Define the Granville–Soundararajan distance $D(f,g;X)$ between two arithmetical functions $f,g:\mathbb{N}\to\mathbb{C}$ with $|f(p)|=|g(p)|=1$ for all primes p, and show that it is a distance.

State and prove an asymptotic for $D(1,\mu;X)$, where μ is the Möbius function.

Let $f:\mathbb{N}\to\mathbb{C}$ be a multiplicative function supported on the squarefrees, and write

$$\nu=\sum_{n\in\mathbb{N}}\delta_{\log n}f(n)n^{-1-\frac{1}{\log X}}.$$

Show that if $|\hat{\nu}(t)|\geqslant\delta\log X$ then $D(f,n^{it};X)=O_\delta(1)$.

3

Define the Riemann ζ-function $\zeta(s)$ for $\Re s>1$. Show that it admits a meromorphic extension to $\Re s>0$, analytic except for a simple pole at $s=1$. Define the Γ-function and state the functional equation for ζ. What is meant by the *critical strip*? Show that ζ has zeros at $-2,-4,-6,\ldots$ and nowhere else outside the critical strip.

[You may assume that $z\Gamma(z)=\Gamma(z+1)$ for $\Re z>0$ without proof, as well as the fact that $\Gamma(z)\neq 0$.]

Prove that the number of zeros of $\zeta(s)$ in the critical strip with imaginary part of magnitude at most T ($T\geqslant 2$) is $O(T\log T)$.

[You may use Jensen's formula without proof.]

Show that ζ has no zeros on the line $\Re s=1$.

4

"If a bounded function $f : \mathbb{N} \to \mathbb{C}$ correlates with the Möbius function then f is either somewhat periodic or somewhat multiplicative". State a precise version of this principle.

Show that

$$\lim_{X\to\infty} \frac{1}{X}\Big|\sum_{n\leqslant X} \mu(n)e(n\sqrt{2})\Big| = 0.$$

5

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\phi(0) = 1$ and $\phi(x) = 0$ for $|x| \geqslant 1/3$ (you may assume without proof that such a function exists). Explain why the function

$$F(n) = \big(\sum_{d|n} \mu(d)\phi(\frac{\log d}{\log X})\big)^2$$

satisfies $F(p) = 1$ whenever $p > X^{1/3}$ is a prime.

Define a function ψ by

$$e^x\phi(x) = \int_{-\infty}^{\infty} \psi(t)e^{-ixt}dt.$$

Show that for any fixed $A > 0$ we have the estimate $|\psi(t)| \ll_A |t|^{-A}$ for all $|t| \geqslant 1$ [you may assume the Fourier inversion formula].

Find a simple expression for

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t)\psi(t')\frac{(1+it)(1+it')}{2+it+it'}dtdt'.$$

Hence, or otherwise, give an asymptotic as $X \to \infty$ for

$$\sum_{n\in I} F(n),$$

for any interval $I \subseteq \mathbb{N}$ of length X.

[You should give no more than the basic structure of the proof, to the point where an educated reader could reasonably easily fill in, or at least believe, the details.]

Show that

$$\pi(X_0 + X) - \pi(X_0) \ll \frac{X}{\log X}$$

whenever $X_0, X \geqslant 2$.

END OF PAPER