UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 1:30 pm to 4:30 pm

PAPER 28

LOCAL FIELDS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove a classification of the non-archimedean absolute values on $\mathbb{Q}$.

(b) Let $(K, |\cdot|)$ be a non-archimedean locally compact valued field, and let $t \in K$ be an element satisfying $0 < |t| < 1$.

(i) Show there exists a finite subset $A \subset K$ such that every element of K can be written uniquely as a Laurent power series in t with coefficients in A.

(ii) Show that if $\mathrm{char}(K) = 0$ then K is a finite extension of $\mathbb{Q}_p$ for some prime p.

(iii) Give an example of a field that is complete with respect to a discrete valuation but is not locally compact. Justify your answer.

2

(a) State and prove a version of Hensel's lemma.

(b) Determine the number of roots of $f(X) = X^3 - 5X + 20$ in $\mathbb{Z}_p$ for $p = 2, 3, 5$.

(c) Let $K = \mathbb{Q}(\alpha)$ where α is a root of f. Show that the different of K is $\mathfrak{p}^2\mathfrak{q}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are ideals in $\mathcal{O}_K$ of norms 5 and 103.

[The discriminant of $X^3 + aX + b$ is $-4a^3 - 27b^2$. General facts about the different may be quoted without proof provided you state them clearly.]

3

(a) Prove that if non-trivial absolute values $|\cdot|_1$ and $|\cdot|_2$ on a field K induce the same topology then there exists $c > 0$ such that $|x|_2 = |x|_1^c$ for all $x \in K$.

(b) Let K be a number field and S a finite set of places of K.

(i) Define the group of idèles J_K. Show that if $x \in J_K$ and $\varepsilon > 0$ then there exists $y \in K$ such that $|x_v - y|_v < \varepsilon$ for all $v \in S$.

(ii) Show that there is a quadratic extension L/K such that every (finite) prime $\mathfrak{p}$ in S is inert in L (i.e. $\mathfrak{p}\mathcal{O}_L$ is a prime).

4

(a) Show that an extension of p-adic fields L/K is totally ramified if and only if $L = K(\alpha)$ where α is a root of an Eisenstein polynomial.

(b) State and prove Krasner's lemma.

(c) Show that there are only finitely many extensions of K of any given degree. Determine the number of quadratic extensions of K in the case p is odd.

[Results about unramified extensions may be quoted without proof.]

5

Write an essay on

EITHER : The Herbrand quotient and its role in norm index calculations for L/K a cyclic extension of p-adic fields.

OR : The Hilbert norm residue symbol and the Hasse-Minkowski theorem.

END OF PAPER