UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 1:30 pm to 4:30 pm

PAPER 26

INTRODUCTION TO IWASAWA THEORY

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Notation: *If p is a prime number, $\mathbb{Z}_p$ will denote the ring $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. If K/F is a Galois extension of fields, $\mathrm{Gal}(K/F)$ will denote the Galois group of K over F.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let p be any prime number. Prove that there exists a Galois extension $\mathbb{Q}[p^\infty]$ of $\mathbb{Q}$ so that $\mathrm{Gal}(\mathbb{Q}[p^\infty]/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p$. For each integer $n \geqslant 0$, let $\mathbb{Q}[p^n]$ denote the unique subfield of $\mathbb{Q}[p^\infty]$ of degree p^n over $\mathbb{Q}$, and write $h(p^n)$ for the class number of $\mathbb{Q}[p^n]$. If $n \leqslant m$, prove that $h(p^n)$ divides $h(p^m)$. [You may assume that p is totally ramified in $\mathbb{Q}[p^n]$ for all $n \geqslant 1$.]

2

Let K be a finite extension of $\mathbb{Q}$, and p any prime number. Assume that K_∞ is any Galois extension of K such that $\mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbb{Z}_p$. Prove that at least one finite prime of K must ramify in K_∞. Prove also that if v is any finite prime of K which does ramify in K_∞, then v must divide p. Finally, if K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, prove that every prime of K dividing p must ramify in K_∞.

3

Let F be a finite extension of $\mathbb{Q}$, and F_∞/F a Galois extension with $\Gamma = \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p$. Let γ be a topological generator of Γ. Assume that M_∞ is a Galois extension of F such that $X = \mathrm{Gal}(M_\infty/F_\infty)$ is abelian and pro-p. Define the natural structure of X as a module over the Iwasawa algebra of Γ. Let M be the maximal abelian extension of F contained in M_∞. Prove that

$$\mathrm{Gal}(M/F_\infty) = X/(\gamma - 1)X\,.$$

4

Write an essay on <u>one</u> of the following two topics:-

(i). Let p be an odd prime number, and let F_∞ be the field obtained by adjoining all p-power roots of unity to $\mathbb{Q}$. Put $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$. Let M_∞ be the maximal abelian p-extension of F_∞, which is unramified outside of p. Put $X = \mathrm{Gal}(M_\infty/F_\infty)$. Describe what is known about the structure of X as a Galois module over the Iwasawa algebra of G, and indicate how these results are proven.

(ii). Let F be any finite extension of $\mathbb{Q}$, and F_∞/F any Galois extension with $\Gamma = \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p$. Let L_∞ be the maximal unramified abelian p-extension of F_∞, and put $Y = \mathrm{Gal}(L_\infty/F_\infty)$. Sketch the proof that Y is always a torsion module over the Iwasawa algebra of Γ.

END OF PAPER