UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2012 1:30 pm to 4:30 pm

PAPER 25

CATEGORY THEORY

*Attempt no more than **FIVE** questions.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\mathcal{C}$ be a locally small category. Define the Yoneda embedding $Y : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathrm{Set}]$. State and prove the Yoneda Lemma.

(b) Let $F, G \colon \mathcal{C} \to \mathrm{Set}$ be functors. Use the Yoneda Lemma to show that a natural transformation $\alpha \colon F \to G$ is a monomorphism in $[\mathcal{C}, \mathrm{Set}]$ if and only if all components α_A are monomorphisms in Set.

2

(a) Prove that if a category has equalisers and finite products, then it has all finite limits.

(b) Deduce that if a category has a terminal object and pullbacks, then it has all finite limits.

(c) Show that if $\mathcal{C}$ has pullbacks, then each slice category $\mathcal{C}/A$ is finitely complete.

3

(a) Define the terms *monomorphism*, *epimorphism*, *strong monomorphism*, *regular monomorphism* and *balanced category*. Show that a regular monomorphism is indeed a monomorphism. Show that every regular monomorphism is strong.

(b) If every monomorphism in $\mathcal{C}$ is strong, show that $\mathcal{C}$ is balanced. Conversely, if $\mathcal{C}$ is balanced and has pullbacks, show that every monomorphism in $\mathcal{C}$ is strong.

[You may assume the result that pullbacks preserve monomorphisms.]

4

State and prove the Special Adjoint Functor Theorem.

[You may use standard results from the course provided they are stated clearly.]

5

Consider a functor $G\colon \mathcal{C} \to \mathrm{Set}$ with $\mathcal{C}$ locally small and complete.

(a) Define, for each object $X \in \mathrm{Set}$, the category $(X \downarrow G)$.

(b) Show that G is representable if and only if $(1 \downarrow G)$ has an initial object.

(c) Show that a complete, locally small category has an initial object if and only if it has a weakly initial set.

(d) Deduce that $G\colon \mathcal{C} \to \mathrm{Set}$ is representable if and only if it preserves limits and $(1 \downarrow G)$ has a weakly initial set.

[You may assume results about limits in $(X \downarrow G)$, provided they are stated clearly. You may also assume that representable functors preserve limits.]

6

(a) Define the structure of a monad on a category $\mathcal{C}$, and the category of algebras $\mathcal{C}^{\mathbb{T}}$ for a monad $\mathbb{T}$. Show that any adjunction $F \dashv G$, where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, induces a monad on $\mathcal{C}$.

(b) Prove that there is an adjuction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ with $F^{\mathbb{T}}\colon \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ and $G^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ inducing the monad $\mathbb{T}$.

7

(a) Define the notions of *semi-additive* and *preadditive* category, and show that finite products and coproducts coincide in a semi-additive category.

(b) Recall that, in any category, a parallel pair $f, g\colon A \to B$ is *reflexive* if there exists $r\colon B \to A$ such that $fr = gr = 1_B$. Show that any reflexive pair (f, g) in a preadditive category $\mathcal{C}$ has the structure of an *internal groupoid*: that is, for each object C, the set $\mathcal{C}(C, B)$ is the set of objects of a groupoid whose morphisms are the elements of $\mathcal{C}(C, A)$, with "domain" and "codomain" given by composition with f and g respectively.

8

(a) Recall that a *pseudo-epimorphism* is a morphism $g \colon A \to B$ such that $fg = 0$ implies $f = 0$. Let $\mathcal{C}$ be pointed with kernels and cokernels such that every monomorphism in $\mathcal{C}$ is normal. Prove that every morphism of $\mathcal{C}$ factors as a pseudo-epimorphism followed by a monomorphism.

(b) Define image factorisation in an abelian category. State the Short Five Lemma. Show how to use image factorisation to deduce the Five Lemma from the Short Five Lemma.

END OF PAPER