UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 9:00 am to 12:00 pm

PAPER 23

TOPOS THEORY

*Attempt no more than **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In $[M^{\mathrm{op}}, \mathbf{Set}]$ for M a monoid observe that an object X is a right action $X \times M \to X$ of M on a set X and that, Y being another object, $Hom(X, Y)$ is the set of equivariant maps $e : X \to Y$ [maps with $e(xm) = (ex)m$ for all $x \in X$, $m \in M$]. Prove that the exponential Y^X in $[M^{\mathrm{op}}, \mathbf{Set}]$ is the set $Hom(M \times X, Y)$ of equivariant maps $e : M \times X \to Y$, where M is the set M with right action by M, with the action $e \to ek$ of $k \in M$ on e defined by $(ek)(g, x) = e(kg, x)$.

(b) For objects X, Y in $[G^{\mathrm{op}}, \mathbf{Set}]$, for G a group, show that the exponential Y^X can be described as the set of all functions $f : X \to Y$, with the right action of $g \in G$ on such a function defined by $(fg)x = [f(xg^{-1})]g$ for $x \in X$.

2

For any set T, the constant presheaf T on a space X has $T(U) = T$ for all open sets U in X, with all restriction maps the identity. Show, using germs, that the associated étale bundle is the projection $p : X \times T \to X$ of the product, where T has the discrete topology; conclude that the associated sheaf is the "constant" sheaf Δ_T, for which $\Delta_T(V)$ is the set of all locally constant functions $V \to T$. Prove also that this defines a functor $\Delta : \mathbf{Set} \to \mathbf{Sh}(X)$ which is left adjoint to the global sections functor $\mathbf{Sh}(X) \to \mathbf{Set}$.

3

Sketch the proof that every functor $f : \mathcal{C} \to \mathcal{D}$ between small categories $\mathcal{C}$ and $\mathcal{D}$ induces an essential geometric morphism $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

4

Sketch the proof that every Grothendieck topos is an elementary topos and give an explicit description of the Heyting algebra operations on its subobject lattices.

5

(a) Show that every Grothendieck topos admits a unique (up to isomorphism) geometric morphism to **Set**.

(b) A Grothendieck topos $\mathcal{E}$ is said to be *local* if the direct image of the unique geometric morphism $\mathcal{E} \to \mathbf{Set}$ is also an inverse image functor. What can you say about a small category $\mathcal{C}$ (resp. a topological space X) if $[\mathcal{C}, \mathbf{Set}]$ (resp. $\mathbf{Sh}(\mathrm{X})$) is a local topos?

6

(a) Introduce the notion of geometric theory and sketch the construction of classifying toposes via syntactic sites.

(b) Is the classifying topos of a Horn theory always (equivalent to) a presheaf topos?

(c) Explain the sense in which classifying toposes can act as 'bridges' for transferring information between Morita-equivalent theories [You might wish to mention the duality theorem].

END OF PAPER