UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Thursday, 7 June, 2012   9:00 am to 12:00 pm

PAPER 22

COMPLEX MANIFOLDS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let V be a codimension r complex submanifold of an n-dimensional complex manifold M. Define what is meant by the *holomorphic normal bundle* N_V of V in M, and prove the adjunction formula for the canonical bundle of V, namely that

$$K_V \cong K_M|_V \otimes \bigwedge^r N_V.$$

Suppose now that V is of codimension 1; define the line bundle $[V]$ on M and prove that $N_V \cong [V]|_V$.

Suppose that a holomorphic vector bundle $E = L_1 \oplus \ldots \oplus L_r$ is a sum of holomorphic line bundles on M, and that V is a codimension r submanifold of M defined by the vanishing of some global holomorphic section of E; prove that $N_V \cong E|_V$. Find a formula for the canonical bundle K_V.

Consider now $M = \mathbf{P}^n$, and let $[H]$ denote the hyperplane line bundle, with dual bundle $[-H]$. By convention, $[aH]$ denotes $[H]^{\otimes a}$ if $a \geqslant 0$, and $[-H]^{\otimes -a}$ if $a \leqslant 0$. Show that $K_M \cong [-(n+1)H]$. Suppose that V is a codimension r submanifold of M defined by the vanishing of homogeneous polynomials $F_1, \ldots, F_r$, where F_i is homogeneous of degree $d_i > 0$. Deduce an expression for K_V.

2

Define what is meant by a *connection* D on a complex C^∞ vector bundle E over a complex manifold M, and also the *curvature* Θ associated with D. For E a complex line bundle, show that Θ is a globally defined closed 2-form. State and prove
(a) the change of frame formula for the connection matrix, and
(b) Cartan's equation relating the curvature and connection matrices.

For E a complex line bundle, explain how $c_1(E) \in H^2(M, \mathbf{Z})$ may be defined by means of the cohomology sequence associated to a certain short exact sequence of sheaves. Identifying the sheaf cohomology group $H^2(M, \mathbf{C})$ with the De Rham cohomology group $H^2_{DR}(M, \mathbf{C})$ via De Rham's Theorem, show explicitly that the class of $\frac{i}{2\pi}\Theta$ determines an integral class (that is, one in the image of $H^2(M, \mathbf{Z})$ in $H^2(M, \mathbf{C})$), independent of the choice of connection D. [Standard properties of Čech cohomology on manifolds may be assumed.]

Define the Fubini–Study form ω on $\mathbf{P}^n(\mathbf{C})$, briefly justifying why its class in $H^2(M, \mathbf{C})$ is integral (you need not check the fact that it determines a hermitian inner-product on the tangent spaces).

3

Let E be a complex vector bundle on a manifold M and D be a connection on E. If D_1, D denote two connections on E, show that $D_1 - D \in \mathcal{A}^1_M(\mathrm{Hom}(E,E))$, the sheaf of smooth 1-forms with coefficients in the bundle $\mathrm{Hom}(E,E)$. Given any connection D on E, show that there is an associated connection $\tilde{D}$ on $\mathrm{Hom}(E,E)$ given by the formula

$$(\tilde{D}\phi)(s) = D(\phi(s)) - \phi(D(s))\,,$$

for ϕ a local section of $\mathrm{Hom}(E,E)$ and s a local section of E.

If we now write $D_1 = D + a$ with $a \in A^1(\mathrm{Hom}(E,E))$, with Θ (respectively Θ_1) denoting the curvature of E determined by D (respectively D_1), prove that

$$\Theta_1 = \Theta + \tilde{D}(a) + a \wedge a,$$

where the last two terms on the right should be carefully defined.

Suppose now that E is a holomorphic vector bundle over a complex manifold M. For a given choice of hermitian metric on E, state the defining properties of the *Chern connection* D, and prove both existence and uniqueness of the connection with these properties. Prove that the corresponding curvature $\Theta \in A^{1,1}(\mathrm{Hom}(E,E))$, and that $\bar{\partial}_{\mathrm{Hom}(E,E)}\Theta = 0$. Without doing the calculations, explain how you would verify that the connection $\tilde{D}$ defined above on $\mathrm{Hom}(E,E)$, corresponding to a Chern connection D on E, is itself a Chern connection.

If D and $D+a$ represent the Chern connections of E with respect to different choices of the hermitian metric on E, show that $a \in A^{1,0}(\mathrm{Hom}(E,E))$ and that

$$\Theta_{D+a} = \Theta_D + \bar{\partial}_{\mathrm{Hom}(E,E)}(a).$$

Hence deduce that Θ determines a well-defined class $\alpha(E)$ in the Dolbeault cohomology group $H^1(M, \Omega^1_M(\mathrm{Hom}(E,E)))$, independent of the choice of metric. For each $k \geqslant 1$, show moreover that $\alpha(E)$ determines a class $\alpha(E)^k \in H^k(M, \Omega^k_M(\mathrm{Hom}(E,E)))$, independent of any choice of the metric.

[Standard facts about cohomology may be assumed in this question, as may Cartan's equation relating the connection and curvature matrices.]

4

Given a hermitian metric on an n-dimensional complex manifold M, and a hermitian metric on a complex vector bundle E over M, explain how an associated hermitian metric $(\ ,\ )$ is induced on the bundle $\bigwedge^p (T'_M)^* \otimes \bigwedge^q (T''_M)^* \otimes E$. Describe the defining property for the Hodge operator $*_E : \mathcal{A}_M^{p,q}(E) \to \mathcal{A}_M^{n-p,n-q}(E^*)$ on the sheaf of smooth (p,q)-forms with coefficients in E. If ω represents the $(1,1)$-form associated to the metric on M, we define an operator (on sheaves) $L : \mathcal{A}_M^{r,s}(E) \to \mathcal{A}_M^{r+1,s+1}(E)$ given by exterior product with ω, and we define $\Lambda : \mathcal{A}_M^{p,q}(E) \to \mathcal{A}_M^{p-1,q-1}(E)$ to be given by $\Lambda = (-1)^{p+q} *_{E^*} L *_E$. Show that Λ is the pointwise adjoint of L, that is $(L\psi(z), \eta(z))_z = (\psi(z), \Lambda\eta(z))_z$ at all points $z \in M$, where $\psi(z)$, respectively $\eta(z)$, are elements of the fibre at z of $\bigwedge^p (T'_M)^* \otimes \bigwedge^q (T''_M)^* \otimes E$, respectively $\bigwedge^{p+1} (T'_M)^* \otimes \bigwedge^{q+1} (T''_M)^* \otimes E$. [Standard properties of the ordinary Hodge $*$-operator may be stated without proof.]

Suppose that E is also a holomorphic bundle; describe the operators $\bar\partial_E$ and $\bar\partial_E^*$; in the case when M is compact, show that the induced operators on global sections $A^{p,q}(E)$, respectively $A^{p,q+1}(E)$, are *adjoint* with respect to suitably defined global inner-products. Letting

$$\Delta_{\bar\partial} = \bar\partial_E \bar\partial_E^* + \bar\partial_E^* \bar\partial_E$$

be the $\bar\partial$-Laplacian acting on global sections, deduce that the commutator $[\Delta_{\bar\partial}, L]$ acting on $A^{p,q}(E)$ has adjoint operator $[\Lambda, \Delta_{\bar\partial}]$ acting on $A^{p+1,q+1}(E)$. [The *commutator* of two operators $[A, B] := AB - BA$.]

Let D denote the Chern connection D on E and let D' denote its $(1,0)$-component. Assuming M is compact, give (without proof) an explicit description for the adjoint $(D')^*$ to D'.

Suppose now that M is compact and that the given metric on M is Kähler. Show that the commutator $[\Lambda, \bar\partial_E^*]$ acting on $A^{p,q}(E)$ is zero. Deduce the Hodge identity for the commutator $[\Lambda, \bar\partial_E]$ on $A^{p,q}(E)$ from a corresponding Hodge identity on $A^{p,q}$, which should be clearly stated. Hence, or otherwise, show that the operator $[L, \Delta_{\bar\partial}]$ acting on $A(E) = \Gamma(M, \mathcal{A}_M(E))$ is zero if and only if the curvature Θ of the Chern connection D vanishes identically. [You may assume that the curvature Θ of D is of type $(1,1)$.]

END OF PAPER