UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 9:00 am to 12:00 pm

PAPER 21

GALOIS COHOMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let K be a field and let A be a central simple algebra over K.

1. Give the definition of the reduced norm $\mathrm{Nrd} : A \to K$ and show that it does not depend on choices.

2. Define the C_r-property for fields and state the theorems of Chevalley, Lang and Tsen respectively, giving examples of C_1-fields [no proofs are required]. Let L/K be a finite extension of fields. Show that if K is C_1, then L is C_1.

3. Prove that if K is C_1, then $Br(K) = 0$.

4. If K is the function field of a curve over a finite field, show that A is split by a cyclic extension of K.

5. Let K be a C_2-field and let D be a central division algebra over K. Show that $\mathrm{Nrd} : D \to K$ is surjective.

2

Let K be a (non-archimedean) complete local field with perfect residue field k. State Witt's theorem [no proof is required].

From now on, k is a finite field. Define explicitly the invariant map $\mathrm{inv}_K : Br(K) \to \mathbf{Q}/\mathbf{Z}$ and show that it is an isomorphism [you may quote without proof any results from the lectures concerning the cohomology of finite fields and the Brauer group of the maximal unramified extension of K].

Let L/K be a finite field extension. Show that $\mathrm{inv}_K \circ \mathrm{Cor}_{L/K} = \mathrm{inv}_L$ and $\mathrm{inv}_L \circ \mathrm{Res}_{L/K} = [L : K] \cdot \mathrm{inv}_K$.

How many division algebras central over K of degree n are there, up to isomorphism? [You may quote any results from the lectures concerning division algebras and their splitting fields.]

3

Let L/K be a finite Galois extension. State and prove Galois descent for finite-dimensional L-vector spaces equipped with a semi-linear action of $\mathrm{Gal}(L/K)$.

Let $\mathrm{CSA}_n(L/K)$ be the set of isomorphism classes of central simple algebras over K of degree n split by L. Define a natural map $\mathrm{CSA}_n(L/K) \to H^1(\mathrm{Gal}(L/K), \mathrm{Aut}_L(M_n(L)))$ and prove it is an isomorphism.

4

In this question, any results from the lectures can be assumed without proof as long as they are clearly stated.

1. Let K be a field of characteristic $p > 0$. Show that $Br(K)$ is p-divisible, i.e. that for all $x \in Br(K)$ there exists a $y \in Br(K)$ such that $x = py$.

2. Let now K be a field of characteristic $\neq p$ and let n be a positive integer. Let K_s denote a separable closure of K. Show that the following statements are equivalent:

(i) $\mathrm{cd}_p(K) \leqslant n$

(ii) For all algebraic extensions L/K, we have $H^{n+1}(L, K_s^\times)\{p\} = 0$ and $H^n(L,, K_s^\times)$ is p-divisible.

(iii) For all finite extensions L/K of degree prime to p, we have $H^{n+1}(L, K_s^\times)\{p\} = 0$ and $H^n(L,, K_s^\times)$ is p-divisible.

Here, for A a torsion abelian group, $A\{p\}$ denotes the p-primary torsion subgroup of A. It might be useful to introduce the Galois module μ_p of p^{th}-roots of unity in $K_s^\times$.

END OF PAPER