UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 1:30 pm to 4:30 pm

PAPER 20

HAMILTONIAN DYNAMICAL SYSTEMS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider $\mathbb{R}^{2n}$ with the standard symplectic form.

i) Define a *canonical transformation* $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$.

ii) Define the Poisson bracket $\{\cdot, \cdot\}_{(q,p)}$. Let

$$\begin{aligned} f : \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \\ (q,p) &\longmapsto (Q,P) \end{aligned}$$

be a diffeomorphism. Prove that the following properties are equivalent:

a) f is a canonical transformation;

b) $\forall\, F, G \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, one has

$$\{F, G\}_{(Q,P)} \circ f = \{F \circ f, G \circ f\}_{(q,p)}\,;$$

c) $\{Q_i, Q_j\}_{(q,p)} = \{P_i, P_j\}_{(q,p)} = 0$ and $\{Q_i, P_j\}_{(q,p)} = \{q_i, p_j\}_{(q,p)}$ for all $i, j = 1, \ldots, n$.

iii) Let $Q = a(q,p)$ be a C^2 transformation from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$, such that $\operatorname{rank}\left(\frac{\partial a}{\partial q}, \frac{\partial a}{\partial p}\right) = n$. Provide (with proof) sufficient and necessary conditions to extend a to a canonical transformation in a neighbourhood of any point, *i.e.* to extend it locally to $(Q,P) = f(q,p)$, where $f_i = a_i$ for all $i = 1, \ldots, n$.

2

Let (M, ω) be a symplectic manifold and let $H : M \longrightarrow \mathbb{R}$ be a C^2 Hamiltonian.

i) Define an *integral of motion* of H. Let F be a C^2 integral of motion of H. Is it true that H is an integral of motion of F? Prove that the corresponding Hamiltonian vector fields X_F and X_H commute.

ii) Consider on $\mathbb{R}^4$ the Hamiltonian system given by

$$H(x_1, x_2, y_1, y_2) = \frac{1}{2}\big(y_1^2 + y_2^2\big) + \frac{1}{2}\big(\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2\big),$$

with $\alpha_1, \alpha_2 > 0$. Find two integrals of motion, which are in involution and are functionally independent on $\mathbb{R}^4 \setminus \big(\{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}\big)$.
If $\dfrac{\alpha_2}{\alpha_1}$ is irrational, prove that there cannot be further independent integrals of motion.
[*Hint: Consider the change of coordinates*

$$(\sqrt{\alpha_i}x_i, \frac{1}{\sqrt{\alpha_i}}y_i) = (\sqrt{2R_i}\sin\theta_i, \sqrt{2R_i}\cos\theta_i) \qquad i = 1, 2. \quad]$$

iii) Consider a Hamiltonian $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ with r integrals of motion: $G_1 = H, G_2, \ldots, G_r$. For $c = (c_1, \ldots, c_r) \in B_\rho(0) \subset \mathbb{R}^r$, let us consider the common level sets

$$\Sigma_c := \{G_1 = c_1, \ldots, G_r = c_r\}$$

and suppose that $dG_1, \ldots, dG_r$ are linearly independent on Σ_c for $c \in B_\rho(0)$. Prove that there exists a Hamiltonian K such that $X_K\big|_{\Sigma_0} = X_H\big|_{\Sigma_0}$, but $X_K\big|_{\Sigma_c} \neq X_H\big|_{\Sigma_c}$ for all other $c \in B_\rho(0) \setminus \{0\}$.

3

Let (M, ω) be a symplectic manifold.

i) Define a *completely integrable* Hamiltonian system and provide an example (with justification).

ii) State and prove the Liouville–Arnol'd theorem on the integrability of Hamiltonian systems.

4

i) Let $\omega \in \mathbb{R}^n$. Explain what it means for ω to be (γ, τ)-Diophantine, with $\gamma > 0$ and $\tau \geqslant n-1$.
Let $\mathcal{H}(\mathbb{T}^n_\sigma)$ be the set of real-analytic functions $f : \mathbb{T}^n \to \mathbb{R}$ with holomorphic extension on $\mathbb{T}^n_\sigma := \{x \in \mathbb{C}^n : \; |\mathrm{Im}\, x_j| \leqslant \sigma, \; \mathrm{Re}\, x_j \in \mathbb{T} \text{ for } j = 1, \ldots, n\}$, with norm $|f|_\sigma = \sup_{\mathbb{T}^d_\sigma} |f|$. Let us consider the differential operator $\mathcal{D}_\omega := \omega_1 \frac{\partial}{\partial x_1} + \ldots + \omega_n \frac{\partial}{\partial x_n}$. Under which conditions on $g \in \mathcal{H}(\mathbb{T}^n_\sigma)$, does there exist a solution to the cohomological equation $\mathcal{D}_\omega f = g$? Is this solution unique? Prove that for each $0 < \delta < \sigma$, f is in $\mathcal{H}(\mathbb{T}^n_{\sigma-\delta})$ and provide an upper bound for $|f - \langle f \rangle|_{\sigma-\delta}$, where $\langle f \rangle$ denotes the average of f on $\mathbb{T}^n$.

ii) Consider the Hamiltonian $H(x, y) = \omega \cdot y + \frac{1}{2}\|y\|^2 + \epsilon V(x)$ defined on $\mathbb{T}^n \times B^n$, where $B^n \subset \mathbb{R}^n$ is an open ball centered at the origin. Assume that ω is (γ, τ)-Diophantine, and V is analytic. Prove that if ϵ is sufficiently small, there exists a symplectic transformation $(x, y) = \Phi(x', y')$ close to the identity, such that

$$H \circ \Phi(x', y') = E_\epsilon + \omega \cdot y' + \frac{1}{2} Q_\epsilon(x', y') + \epsilon^2 \tilde{V}(x)$$

where $E_\epsilon = O(\epsilon)$ and Q_ϵ is quadratic in y' and ϵ-close to $\|y'\|^2$.

END OF PAPER