UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 9:00 am to 12:00 pm

PAPER 19

ALGEBRAIC TOPOLOGY

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the degree of a map $f : S^n \to S^n$.

The suspension SX of a space X is $(X \times [0,1])/\sim$, where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$.

Show that, if X is a compact manifold, then SX is a manifold if and only if X is a sphere.

Show that for every m and n, there is a map $f : S^n \to S^n$ of degree m.

2

Define the relative homology groups $H_n(X, A)$, where A is a subspace of the space X.

Show that, under circumstances which are to be stated, $H_n(X, A)$ is isomorphic to $\tilde{H}_n(X/A)$.

Suppose that X is a compact oriented surface of genus 2 and that A is a simple closed curve in X. Calculate $H_*(X/A)$ in the following cases.

(1) A cuts X into 2 pieces, each a genus 1 surface with a disc removed;

(2) cutting X along A gives a genus 1 surface with 2 discs removed.

3

Describe, in outline, the construction of cellular homology. Show that $\mathbf{P}^2_{\mathbf{C}}$ and $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ are not homeomorphic.

Show, by considering copies of $\mathbf{P}^1_{\mathbf{C}}$ in $\mathbf{P}^2_{\mathbf{C}}$, that the intersection pairing $H_2(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z}) \times H_2(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z}) \to \mathbf{Z}$ is unimodular.

4

Suppose that A is a commutative ring that is an $\mathbf{R}$-vector space. A non-degenerate dot product on A is a positive definite symmetric $\mathbf{R}$-bilinear form $b : A \times A \to \mathbf{R}$, written as $b(x, y) = x \cdot y$, such that $|xy| = |x|\,|y|$, where $|x| = (x \cdot x)^{1/2}$, for all $x, y \in A$.

Suppose that A has a non-degenerate dot product and that $\dim_{\mathbf{R}} A = n$ is finite. By considering the squaring map, or otherwise, show that $n \leqslant 2$.

[You may assume the result that if $g : M \to N$ is a differentiable map of manifolds such that g is injective and the derivative $g_* : T_m M \to T_{g(m)} N$ is injective for all $m \in M$, and if M is compact, then g is a closed embedding.]

END OF PAPER