UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 9:00 am to 12:00 pm

PAPER 18

ABELIAN VARIETIES

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{H}$ be the complex upper half plane and N a positive integer. Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$. For each $\tau \in \mathcal{H}$, we use E_τ to denote the complex torus $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Consider pairs (E, G) where E is a complex torus of (complex) dimension 1 and $G \cong \mathbb{Z}/N\mathbb{Z}$ a subgroup of E. We say that (E_1, G_1) is equivalent to (E_2, G_2) if there is an isomorphism $\varphi : E_1 \to E_2$ such that $\varphi(G_1) = G_2$. We define $\mathscr{E}$ to be set of equivalence classes of the pairs (E, G). Let $\Phi : \mathcal{H} \to \mathscr{E}$ be the natural map that sends $\tau \in \mathcal{H}$ to the pair $(E_\tau, \langle \frac{1}{N} \rangle)$ where $\langle \frac{1}{N} \rangle$ is the cyclic subgroup of E_τ generated by the image of $\frac{1}{N}$.
(i) Show that Φ is surjective.
(ii) Show that $\Phi(\tau_1) = \Phi(\tau_2)$ if and only if $\tau_1 = \gamma \cdot \tau_2$ for some $\gamma \in \Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}$$

Conclude that $\mathscr{E}$ is identified with $\Gamma_0(N)\backslash\mathcal{H}$.

2

Let $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$ be the 1-dimensional complex torus corresponding to $\tau = i \in \mathcal{H}$.
(i) Write down all hermitian forms $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that $E = \mathrm{Im}(H)$ is integral on the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}i$. Show that the group of all such hermitian forms is canonically isomorphic to $\mathbb{Z}$ and the positive definite ones correspond to $\mathbb{Z}_{>0}$.
(ii) Let H_1 be the hermitian form that corresponds to $1 \in \mathbb{Z}$. Describe all $\alpha : \Lambda \to \mathbb{C}_1 = \{z \in \mathbb{Z} : |z| = 1\}$ such that

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E_1(\lambda_1, \lambda_2)} \alpha(\lambda_1)\alpha(\lambda_2),$$

where $E_1 = \mathrm{Im}(H_1)$.
(iii) Let α_1 be one of the above α's with $\alpha_1(1) = \alpha_1(i) = 1$ and $\mathscr{L} = \mathscr{L}(H_1, \alpha_1)$ be the line bundle associated to (H_1, α_1) as in the Theorem of Appell–Humbert. Show that $\mathrm{H}^0(X, \mathscr{L})$ is one dimensional.
(iv) Show that the endomorphism algebra $\mathrm{End}(X)$ is naturally isomorphic to $\mathbb{Z}[i]$.
(v) Let $\varphi \in \mathrm{End}(X)$ correspond to $a + b\,i \in \mathbb{Z}[i]$. If we write $\varphi^*\mathscr{L}$ as $\mathscr{L}(H, \alpha)$, give explicit expressions (in terms of a, b and α_1) for H and α.

3

Let k be an algebraically closed field of characteristic 0. Let A, B be abelian varieties defined over k. For any $f, g \in \mathrm{Hom}(A, B)$ and a line bundle $\mathscr{L}$ on B, we define

$$D_{\mathscr{L}}(f, g) = (f+g)^*\mathscr{L} \otimes f^*\mathscr{L}^{-1} \otimes g^*\mathscr{L}^{-1}.$$

(i) Show that the map $D_{\mathscr{L}} : \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, B) \to \mathrm{Pic}(X)$ is symmetric and bilinear. You may use the theorem of the cube.
(ii) Show that $D_{\mathscr{L}}(f, g) = (f, \phi_{\mathscr{L}} \circ g)^*(\mathscr{P}_B)$, where $\mathscr{P}_B$ is the Poincaré line bundle on $B \times \hat{B}$.
(iii) Show that the map $D_{\mathscr{L}}$ only depends on the class of $\mathscr{L}$ in $\mathrm{NS}(B) = \mathrm{Pic}(B)/\mathrm{Pic}^0(B)$.
(iv) Show that $\phi_{D_{\mathscr{L}}(f,g)} = \hat{f} \circ \phi_{\mathscr{L}} \circ g + \hat{g} \circ \phi_{\mathscr{L}} \circ f$.

4

Let X/k be an abelian variety defined over an algebraically closed field k. This question studies the subgroup of X generated by the differences of points on a curve $C \subset X$.
(i) Let T be a variety and C a smooth projective curve. Given any line bundle $\mathscr{M}$ on $C \times T$, show that the degree $\deg(\mathscr{M}|_{C \times \{t\}})$ does not depend on the point $t \in T$. [*Hint: use the upper semi-continuity theorem.*]
(ii) Let $\varphi : C \hookrightarrow X$ be a smooth projective curve on X and $\mathscr{L}$ a line bundle on X. Show that the degree of $\varphi^* T_x^* \mathscr{L}$ is independent of $x \in X$.
(iii) Let $D \subset X$ be an irreducible divisor that does not meet C. Show that $T^*_{x_1 - x_2} D = D$ for all $x_1, x_2 \in C$.
(iv) Set $\mathscr{L} = \mathcal{O}_X(D)$. Let $Y \subset X$ be the closure of the subgroup generated by $\{x_1 - x_2 : x_1, x_2 \in C\}$. Show that $Y \subset K(\mathscr{L}) = \{x \in X : T_x^*\mathscr{L} \cong \mathscr{L}\}$.
(v) Show that $Y = X$ if and only if C meets all irreducible divisors $D \subset X$. [You may assume the following fact: if $Y \subset X$ is a Zariski closed abelian subgroup, then there is a morphism $f : X \to Z$ such that Y is one of the fibers.]

5

In this question, we work over the field $\mathbb{C}$ of complex numbers. Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth curves of degree 2 (they are also called conics). Assume that C_1 and C_2 meet in 4 distinct points.

(i) Show that the space of lines on $\mathbb{P}^2$, denoted by $(\mathbb{P}^2)^\vee$, is again isomorphic to $\mathbb{P}^2$.

(ii) Let $C_1^\vee \subset (\mathbb{P}^2)^\vee$ denote the space of lines tangent to C_1. Show that $C_1^\vee$ is again a conic.

(iii) Let $E \subset C_1^\vee \times C_2$ consist of all the pairs (l, x) such that $x \in l \subset \mathbb{P}^2$. Show that $E \to C_2$ is a double cover ramified at 4 points. Conclude that E is a smooth curve of genus 1.

(iv) Given $(l_0, x_0) \in E$, then l_0 intersects C_2 in a second point x_1 (the first point is x_0). Through this x_0, there is a second line l_1 tangent to C_1 (the first tangent line is l_0). Hence we get a point $(l_1, x_1) \in E$. Replace (l_0, x_0) by (l_1, x_1) and repeat the above construction. In this way, we get a sequence

$$S_{l_0,x_0} : \{(l_0, x_0), (l_1, x_1), (l_2, x_2), \ldots\}$$

Assume that S_{l_0,x_0} is periodic for some initial point (l_0, x_0). Show that $S_{l,x}$ is periodic for all initial points $(l, x) \in E$. [You may assume any standard results about morphisms between genus 1 curves, as long as they are clearly stated.]

END OF PAPER