UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 1:30 pm to 4:30 pm

PAPER 17

DIFFERENTIAL GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a smooth vector field on a manifold M.

1. Define what it means for a smooth curve $\gamma\colon(-\epsilon,\epsilon)\to M$ to be an *integral curve* of X and
2. Define what is meant by the *flow* $\{\phi_t\}$ of X.

State and prove a uniqueness result for integral curves of X that pass through a given point $p\in M$. Using this, or otherwise, show that if ϕ_t is the flow of X then

$$\phi_{t+s}=\phi_t\circ\phi_s$$

whenever both sides are defined. [Standard results from the theory of Ordinary Differential Equations may be assumed without proof.]

Now suppose that Y is another smooth vector field with flow ψ_s and that the flows of X and Y commute (i.e. $\phi_t\circ\psi_s=\psi_s\circ\phi_t$ whenever both sides are defined). Show that

$$(D_q\phi_t)(Y_q)=Y_{\phi_t(q)}\quad\text{for all } q\in M. \tag{1}$$

Finally suppose that in addition to the flows commuting, the vector fields X and Y are pointwise linearly independent. Prove that given any point $p\in M$ there exists a two dimensional submanifold $S\subset M$ containing p such that if γ is any curve in M with $\gamma(0)=p$ and so $\dot\gamma(t)$ lies in the plane spanned by $X_{\gamma(t)}$ and $Y_{\gamma(t)}$ for all t, then $\gamma(t)\in S$ for all t [Results from lectures may be assumed if stated clearly.]

2

1. Let M be a smooth manifold. Define the space $\Omega^p(M)$ of smooth p-forms and give the defining properties of the exterior derivative map $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$. Prove that such a d exists and is unique.

2. Prove that if $\omega \in \Omega^1(M)$ then

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

for all $X, Y \in \mathrm{Vect}(M)$.

3. Now let W be the space of linear maps $\alpha\colon \mathrm{Vect}(M) \to C^\infty(M)$ such that if $X_p = 0$ then $\alpha(X)(p) = 0$. Prove there exists a natural isomorphism $\theta\colon \Omega^1(M) \to W$ with the property that $\theta(g\alpha) = g\alpha$ for all $g \in C^\infty(M)$.

4. Suppose $H^1_{\mathrm{deRham}}(M) = 0$ and let $\alpha \in W$ be given. Show that there exists a $g \in C^\infty(M)$ such that $\alpha(X) = X(g)$ for all $X \in \mathrm{Vect}(M)$ if and only if $X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) = 0$ for all $X, Y \in \mathrm{Vect}(M)$.

3

Define what is meant by the Lie algebra $\mathfrak{g}$ of a Lie group G, and determine $\mathfrak{g}$ when $G = GL_n(\mathbb{R})$.

Show that for any Lie group G there is a vector space isomorphism between $\mathfrak{g}$ and the space of left invariant vector fields on G.

Now suppose H is another Lie group with Lie algebra $\mathfrak{h}$ and $F\colon G \to H$ is a homomorphism of Lie groups. Show how F induces a morphism $F_*\colon \mathfrak{g} \to \mathfrak{h}$ and prove this is a Lie algebra morphism.

Let $\exp_G\colon \mathfrak{g} \to G$ be the exponential map for G (defined using the flow along invariant vector fields) and similarly $\exp_H$ be the exponential map for H. Prove that $F \circ \exp_G = \exp_H \circ F_*$.

Using this, or otherwise, prove that if A is any real $n \times n$ matrix then

$$\det \exp(A) = e^{\mathrm{trace}(A)}$$

[If you use a specific expression for the exponential map you are expected to prove it].

4

Define what is meant by a *connection* on a vector bundle E. Prove that E admits a connection, and that the space of connections is (non-canonically) isomorphic to $\Omega^1(End(E))$.

Now suppose that ∇ is a linear connection on a manifold M (i.e. a connection on TM). Define a map $\tau\colon \operatorname{Vect}(M) \times \operatorname{Vect}(M) \to \operatorname{Vect}(M)$ by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Show that τ is tensorial (i.e. it is induced by a certain tensor on M whose type you should determine explicitly).

Let $\{e_1, \ldots, e_n\}$ be a local frame for TM over some open set U. Show that there exist unique 1-forms ω_{ij} on U such that

$$\nabla_X e_i = \sum_j \omega_{ij}(X) e_j$$

for all i and all $X \in \operatorname{Vect}(M)$.

Finally let ϕ_i be the frame for T^*M over U that is dual to e_i (i.e. $\phi_i(e_j) = \delta_{ij}$). Prove that

$$d\phi_j = \sum_i \phi_i \wedge \omega_{ij} + \tau_j$$

where

$$\tau(X,Y) = \sum_j \tau_j(X,Y) e_j.$$

5

1. Let M be a manifold with a Riemannian metric g. Given a $\sigma \in C^\infty(M)$ show that there exists a vector field V_σ on M that satisfies

$$g(V_\sigma, Y) = Y(\sigma) \text{ for all } Y \in \mathrm{Vect}(M).$$

Show also that if

$$\tilde{g}(X,Y) = e^{2\sigma} g(X,Y)$$

then $\tilde{g}$ is a well-defined Riemannian metric on M.

2. State the defining properties of the Levi-Civita connection. Denoting the Levi-Civita connection of g (resp. $\tilde{g}$) by ∇ (resp. $\tilde{\nabla}$), prove that

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)V_\sigma$$

where $\tilde{g}$ is as in the first part of the question. Hence or otherwise prove that if M is compact then there exists a point $p \in M$ such that $\tilde{\nabla}_X Y|_p = \nabla_X Y|_p$ for all X and Y.

END OF PAPER