UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 9:00 am to 12:00 pm

PAPER 16

KNOTS AND 4-MANIFOLDS

*Attempt **BOTH** questions from Section I*
*and no more than **THREE** questions from Section II.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

1. Figure 1a is a Kirby diagram representing a 4-manifold W obtained by attaching three 2–handles to a B^4. Find the intersection form on W, the signature of W, and $H_1(\partial W)$.

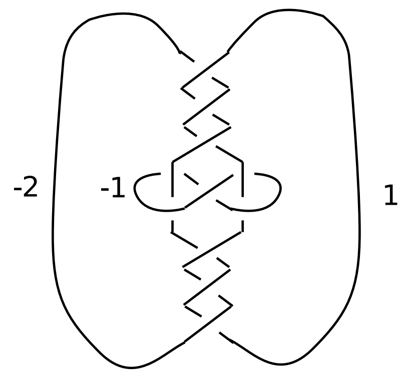

Figure 1a

2. Figure 1b is a genus three Heegaard diagram representing a 3-manifold Y with $\partial Y = T^2$. Find a presentation for $\pi_1(Y)$ and use it to compute $\Delta(Y)$.

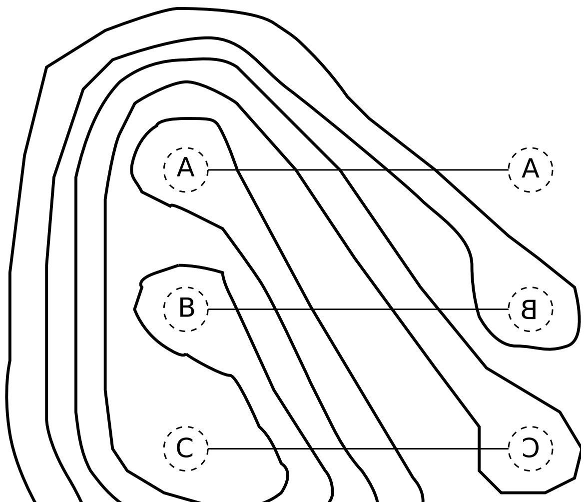

Figure 1b

2

Compute the Alexander polynomial, Seifert genus, signature, and slice genus of the knot shown in Figure 2. (You may use any result from lectures or the example sheets, as long as it is clearly stated.)

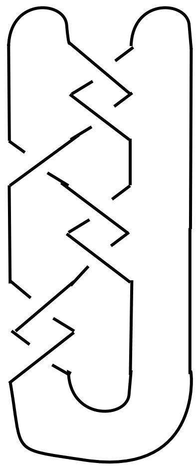

Figure 2

SECTION II

3

Let W^n be a smooth compact n-manifold with boundary. Assuming that W admits a handle decomposition, show that $H_*(W;\mathbb{Z}/2) \cong H^*(W,\partial W;\mathbb{Z}/2)$.

Assuming that the intersection pairing $H_k(M^n;\mathbb{Z}/2) \times H_{n-k}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ is nonsingular for any smooth *closed* manifold M, show that the intersection pairing $H_k(W;\mathbb{Z}/2) \times H_{n-k}(W,\partial W;\mathbb{Z}/2) \to \mathbb{Z}/2$ is nonsingular.

Now suppose that $n = 3$, and let $i_* : H_1(\partial W;\mathbb{Z}/2) \to H_1(W;\mathbb{Z}/2)$ be the map induced by inclusion. Show that $\dim \ker i_* = \frac{1}{2} \dim H_1(\partial W;\mathbb{Z}/2)$. Conclude that $\mathbb{RP}^2$ does not bound a compact 3-manifold.

4

Show that if Y is a closed connected three-manifold, $\pi_1(Y)$ has a presentation with the same number of generators and relations.

Show that if G is a finitely presented group, there is a closed connected 4-manifold M with $\pi_1(M) \cong G$.

5

Let $K \subset S^3$ be a knot. Define the infinite cyclic cover $\widetilde{S^3 - K}$ and explain how it is related to $\Delta_K(t)$.

Suppose $S^3 - K$ is fibred with monodromy $\phi : \Sigma \to \Sigma$. In other words, $S^3 - K \cong \Sigma \times [0,1]/\sim$, where $(\phi(x), 0) \sim (x, 1)$. Show that $\Delta_K(t) \sim \det(tI - \phi_*)$, where $\phi_* : H_1(\Sigma) \to H_1(\Sigma)$ is the induced map.

6

Let K be a knot in S^3. What is a Seifert matrix V for K? Explain (but do not prove) how V is related to $\Delta_K(t)$.

Using the relation you stated above, show that $g(K) \leqslant \deg\Delta_K(t)$.

Draw a picture of a knot $K \subset S^3$ which is not the unknot, but for which $\Delta_K(t) = 1$. Justify your computation of the Alexander polynomial. [You need not justify the fact that your knot is not the unknot.]

7

Define the signature of a knot $K \subset S^3$. Assuming that a 4-manifold which bounds a 5-manifold has signature 0, prove that the signature of K is well-defined.

Show that $\sigma(K) = \sigma(V + V^T)$, where V is a Seifert matrix for K.

If K_1 and K_2 are the two knots shown in Figure 7, show that $\pi_1(S^3 - K_1) \cong \pi_1(S^3 - K_2)$, but $K_1 \ncong K_2$.

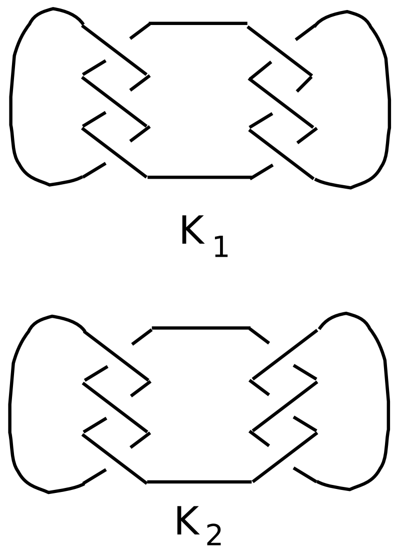

Figure 7

8

Define the Kauffman bracket of a planar knot diagram. Explain how it is related to the Jones polynomial, and use it to prove that the Jones polynomial is invariant under Reidemeister moves.

Show that $|V_L(1)| = 2^l$, where l is the number of components of L.

END OF PAPER