UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2012 1:30 pm to 3:30 pm

PAPER 15

SYMPLECTIC GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let X be a manifold and consider the cotangent bundle $\pi : T^*X \to X$ equipped with its canonical symplectic form $\omega = -d\alpha$, where α is the Liouville 1-form. Let σ be a closed 2-form on X and define

$$\omega_\sigma := \omega + \pi^*\sigma.$$

Show that ω_σ is a symplectic form.

(b) Let θ be a 1-form on X which we also regard as a section $\theta : X \to T^*X$. Show that $\theta(X)$ is a Lagrangian submanifold of (T^*X, ω_σ) if and only if $\sigma = d\theta$. Conclude that if the cohomology class of σ is not zero, then there are no Lagrangian submanifolds L in (T^*X, ω_σ) for which $\pi|_L : L \to X$ is a diffeomorphism.

(c) Assume that σ is exact. Is it true that (T^*X, ω) and (T^*X, ω_σ) are symplectomorphic?

2

Let M be a compact manifold without boundary. Assume that α_t, $t \in [0, 1]$, is a smooth family of contact forms on M. Show that there exists an isotopy $\rho_t : M \times \mathbb{R} \to M$ and a family of smooth nowhere-vanishing functions $u_t : M \to \mathbb{R}$, $t \in [0, 1]$, such that $\rho_t^* \alpha_t = u_t\, \alpha_0$ for all $t \in [0, 1]$.

[*Hint: Search for a time-dependent vector field which belongs to the kernel of* α_t.]

3

Let (M, ω) be a symplectic manifold and let X be a compact Lagrangian submanifold. Let ω_0 denote the canonical symplectic form of T^*X. Show that there are neighbourhoods U_0 of X in T^*X and U of X in M, and a diffeomorphism $\varphi : U_0 \to U$ such that $\varphi^*\omega = \omega_0$ and $\varphi \circ i_0 = i$, where $i_0 : X \to T^*X$ and $i : X \to M$ are the inclusion maps.

[You may assume the relative version of the Moser theorem, provided it is clearly stated.]

4

(a) Let X be n-dimensional manifold and let α denote the Liouville 1-form of T^*X. Given a diffeomorphism $f : X \to X$, explain how to lift it to a natural diffeomorphism $f_\# : T^*X \to T^*X$ such that $f_\#^*\alpha = \alpha$.

(b) Let $g : T^*X \to T^*X$ be a diffeomorphism such that $g^*\alpha = \alpha$. Show that there exists a diffeomorphism $f : X \to X$ such that $f_\# = g$.

(c) Give an example of a symplectomorphism of T^*X which does not preserve the Liouville 1-form α.

END OF PAPER