UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 14

RIEMANNIAN GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the *Ricci curvature* Ric and *sectional curvature* K of a Riemannian manifold M. Prove the relation

$$\sum_{j=1}^{n} \mathrm{Ric}(e_j, e_j) = \sum_{\substack{i,j=1,\dots,n \\ i \neq j}} K(e_i, e_j)$$

at $p \in M$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_pM. Show also that for each tangent vector $X \in T_pM$, the linear map of T_pM given by $v \mapsto R_p(X, v)X$ (where R is the curvature tensor) is self-adjoint.

State the Bonnet–Myers diameter theorem.

What is a left-invariant metric on a Lie group? Show that if G is a compact Lie group with bi-invariant (i.e. left- and right-invariant) metric and the centre of the Lie algebra of G is trivial, then the fundamental group of G is finite.

[Standard properties of the Riemann curvature tensor may be assumed if clearly stated. You may assume that the Levi–Civita connection of a bi-invariant metric satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$, for all left-invariant vector fields X, Y.]

2

Let M be a Riemannian manifold. Give a definition of a *Jacobi field* along a geodesic curve on M.

Show that for every Jacobi field J along a geodesic $\gamma : [0, 1] \to M$ there exists a smooth map $f(t, s)$ of $[0, 1] \times (-\varepsilon, \varepsilon)$ into M such that for any fixed $s \in (-\varepsilon, \varepsilon)$, the curve $\gamma_s(t) = f(t, s)$ is a geodesic, $\gamma_0(t) = \gamma(t)$ and $J(t) = \dfrac{\partial f}{\partial s}(t, 0)$. Deduce from this result a general formula for Jacobi fields J along γ such that $J(0) = 0$.

Define what is meant by the points $p = \gamma(0)$ and $q = \gamma(L)$ being *conjugate* along a geodesic γ. By considering the square length of a Jacobi field $|J(t)|^2$, or otherwise, show that if the sectional curvature is non-positive at each point of γ, then γ has no conjugate points.

[Standard properties of covariant derivative along curves may be assumed if clearly stated.]

3

Define the distance function d induced by the metric on a connected Riemannian manifold. What is a geodesically complete manifold? State the Hopf–Rinow theorem.

Recall that a Riemannian manifold M is said to be *homogeneous* if given any two points p and q in M, there exists an isometry of M taking p to q. Show that a homogeneous Riemannian manifold is complete.

We say that a connected Riemannian manifold M is *two-point homogeneous* if for each $p_1, p_2, q_1, q_2 \in M$, such that $d(p_1, p_2) = d(q_1, q_2)$, there exists an isometry f of M such that $f(p_i) = q_i$ for $i = 1, 2$. Show that M is two-point homogeneous if and only if for each $p, q \in M$ and unit vectors $u \in T_pM$, $v \in T_qM$ there exists an isometry f of M such that $f(p) = q$ and $df_p(u) = v$.

[The Gauss lemma may be assumed.]

4

For an oriented Riemannian manifold (M, g), define the *volume form* ω_g, the *Hodge $*$-operator*, and the *Laplace–Beltrami operator*. State the Hodge decomposition theorem for the space of p-forms.

Explain briefly how a connection on the tangent bundle TM induces a connection on the bundle of differential p-forms on M. Show that the volume form ω_g is parallel, i.e. $\nabla \omega_g = 0$, where ∇ is induced by the Levi–Civita connection of g.

Let M be a compact oriented Riemannian manifold and $\mathcal{H}^p$ the space of harmonic differential p-forms on M. Show that every linear function $f : \mathcal{H}^p \to \mathbb{R}$ $(0 \leqslant p \leqslant \dim M)$ may be expressed as

$$f(\varphi) = \int_M \varphi \wedge \psi$$

for some $(n-p)$-form ψ. Is the form ψ uniquely determined? If not, what is the ambiguity of choosing ψ? Justify your answer.

[You may assume that the formal adjoint of the exterior derivative d on p-forms, $p > 0$, is given by $(-1)^{n(p-1)+1} {*d*}$.]

5

Define the *divergence* of a vector field on a Riemannian manifold. Let (M, g) be a compact oriented Riemannian manifold and ω_g the volume form of g. Given a 1-form θ on M define the vector field X_θ dual to θ with respect to the metric g and prove the identity $\delta\theta = -\operatorname{div} X_\theta$, where δ is the formal adjoint of the exterior derivative.

[You may assume that for each vector field X on M the form $(\operatorname{div} X)\omega_g$ is exact.]

State the Bochner–Weitzenböck formula for 1-forms, explaining carefully all the terms that appear in it.

Suppose that a compact connected Riemannian manifold M has $\mathrm{Ric} \geqslant 0$ at each point. Show that the dimension of the space of harmonic 1-forms on M is not greater than $\dim M$.

END OF PAPER