UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 9:00 am to 12:00 pm

PAPER 13

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

Throughout this paper, k denotes an algebraically closed field.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Give the definition of dimension of a quasi-projective algebraic set. State how this is related to dimension in commutative algebra. Let X be an affine variety of dimension d and $f \in k[X]$ such that $\emptyset \neq V_X(f) \neq X$. Let U be a non-empty open subset of X. Show that $\dim U = \dim X$. Next, prove that every irreducible component of $V_X(f)$ has dimension $d-1$. [Here you are not allowed to use the fact that dimension is equal to the transcendence degree of the function field.]

(ii) Let X be an affine variety, $0 \neq f \in k[X]$, and $U_f := X \setminus V_X(f)$. Show that $k[U_f]$ is isomorphic to the localisation $k[X]_f$ as k-algebras. Assume $X = \mathbb{A}^2_k$ and $W = X \setminus \{(0,0)\}$. Calculate $k[W]$ and deduce that W is not affine.

(iii) Find an affine variety X of dimension 3 and irreducible closed subsets $Y, Z \subset X$ of dimension 2 such that $Y \cap Z$ has dimension 0.

2

(i) Give an example (with justifications) of a regular map $f \colon X \to Y$ of quasi-projective varieties and a point $y' \in Y$ satisfying the following: the fibre X_y is a smooth affine variety of dimension two for every $y \neq y'$ but $X_{y'}$ is singular, not irreducible and not affine.

(ii) Assume $n, m > 0$. Show that $\mathbb{P}^n_k \times \mathbb{P}^m_k$ and $\mathbb{P}^{n+m}_k$ are not isomorphic but they are birationally isomorphic. [If you argue using class groups, then you should carefully calculate these groups.]

3

(i) Let $X = V(F) \subset \mathbb{P}^2_k$ be a projective curve where F is irreducible of degree 3. Show that X has at most 1 singular point.

(ii) Let $k = \mathbb{C}$ and $\mu = e^{\frac{\pi}{3}i}$. Let $\sigma, \tau \colon \mathbb{A}^2_k \to \mathbb{A}^2_k$ be given as $\sigma(a_1, a_2) = (\mu a_1, \mu^{-1} a_2)$ and $\tau(a_1, a_2) = (-a_2, a_1)$ and let G be the subgroup of $\mathrm{Aut}(\mathbb{A}^2_k)$ generated by σ and τ. Show that G is a finite group and show that the quotient of $\mathbb{A}^2_k$ by G is isomorphic to $V(p) \subset \mathbb{A}^3_k$ for some polynomial p.

4

(i) Let X be a normal quasi-projective variety of dimension d. Show that the set of singular points of X is of dimension at most $d-2$.

(ii) Let $k=\mathbb{C}$ and $X=V(t_1t_3-t_2^2)\subset\mathbb{A}_k^3$. Show that X is normal but not smooth. Next, show that there exist a divisor D and a point x on X such that $x\in\mathrm{Supp}D'$ for any divisor $D'\sim D$.

[For a divisor $D=\sum a_iD_i$ with D_i distinct prime divisors, recall that $\mathrm{Supp}D$ is the union of those D_i with $a_i\neq 0$.]

5

(i) State the Riemann–Roch theorem for curves. Next, give its proof using cohomology.

(ii) Give an example (with justifications) of a smooth projective variety X such that $\mathrm{cl}(X)$ is not a finitely generated abelian group. [You need to provide the equations that define X in some projective space.]

END OF PAPER