UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 9:00 am to 12:00 pm

PAPER 12

SPECTRAL GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what is meant by a flat d-dimensional torus. For such a torus describe its spectrum for the Laplacian acting on smooth functions and prove that it is what you say it is.

Prove that isospectral flat two-dimensional tori are isometric.

[You may use results from analysis and lattice theory without proof.]

2

State and prove the transplantation theorem for eigenfunctions of the Laplacians between manifolds constructed from copies of a Euclidean domain by identifying various pairs of boundary faces of those domains.

Construct, with proof, a pair of isospectral surfaces with boundary, one of which is connected with a uniform boundary condition and the other of which comprises two components one of which has uniform, and the other mixed, boundary conditions.

3

Define the heat kernel for a compact Riemannian manifold N. If U is a group of isometries acting freely on N and $M = U \setminus N$ is the quotient manifold with the induced metric, obtain an expression for the heat kernel of M in terms of that of N.

Derive an expression for the heat trace of M and deduce Sunada's theorem concerning isospectral quotients of N.

4

Given a finitely presented group T, show how to construct a manifold N of dimension $d \geqslant 3$ with T acting as a group of isometries without fixed points.

Assuming that T has Gassman equivalent subgroups U_1 and U_2, construct isospectral manifolds covered by N that are not homeomorphic.

For any $n \in \mathbb{N}$ construct 2^n isospectral manifolds no two of which are isometric.

5

Let T be a finite group generated by two of its elements A, B. Describe how to construct a Riemann surface N on which T acts by isometries.

Identify $\chi(N)$ and the non-trivial stabilisers of the action in terms of appropriate properties of A, B and T.

A certain group T has Gassman equivalent subgroups U_1 and U_2 of order 8 and is generated by elements A, B with the following permutation representation, which may be assumed faithful, on the cosets of U_i: A acts as $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$ on the cosets of each of U_1 and U_2; B acts as $(1\ 4\ 10)(2\ 7\ 6)(3\ 8\ 9)(5\ 11\ 12)$ on the cosets of U_1 and as $(1\ 4\ 10)(2\ 7\ 6)(5\ 8\ 9)(3\ 11\ 12)$ on the cosets of U_2. Use these data to construct isospectral Riemann surfaces. Are they isometric? What is their genus?

END OF PAPER