UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 9:00 am to 11:00 am

PAPER 11

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the *shadow* $\partial\mathcal{A}$ of an r-uniform family $\mathcal{A} \subset [N]^{(r)}$, and state the Kruskal–Katona theorem.

Prove that, if $|\mathcal{A}| = \binom{x}{r}$ where $x \in \mathbb{R}$ and $x > r-1$, then $|\partial\mathcal{A}| \geqslant \binom{x}{r-1}$.

A graph G on n vertices and m edges can be viewed as $G \in E^{(m)}$ where $E = [n]^{(2)}$. There are $\binom{N}{m}$ such graphs, where $N = |E| = \binom{n}{2}$. Let $\mathcal{G}_m \subset E^{(m)}$ be the family of those graphs having m edges and containing no triangle. Define x_m by $|\mathcal{G}_m| = \binom{x_m}{m}$, where $x_m \geqslant m-1$. Define $p_m = |\mathcal{G}_m|/|E^{(m)}| = |\mathcal{G}_m|/\binom{N}{m}$.

Prove that

$$\frac{p_{m+1}}{p_m} \leqslant \frac{x_m - m}{N - m}.$$

Infer that

$$p_{m+k} \leqslant p_m \prod_{i=0}^{k-1} \frac{x_m - i}{N - i}$$

and hence that if $p_m \leqslant \frac{1}{2}$ then $p_{2m} \leqslant \frac{1}{4}$.

2

Let p be prime and let $0 \leqslant l_1 < l_2 < \cdots < l_s$. Let $\mathcal{A} \subset \mathcal{P}[n]$ be such that, for $1 \leqslant i \leqslant s$, there is no $A \in \mathcal{A}$ with $|A| \equiv l_i \pmod{p}$, but for all distinct $A, B \in \mathcal{A}$, $|A \cap B| \equiv l_i \pmod{p}$ for some i.

Show that $|\mathcal{A}| \leqslant \sum_{i=0}^{s} \binom{n}{i}$. Show further that, if there is some r with $|A| \equiv r \pmod{p}$ for all $A \in \mathcal{A}$ and $r \not\equiv j \pmod{p}$ for $0 \leqslant j < s$, then $|\mathcal{A}| \leqslant \binom{n}{s}$.

In the latter case, give an example to show that $|\mathcal{A}| > \binom{n}{s}$ is possible if the condition $r \not\equiv j \pmod{p}$ for $0 \leqslant j < s$ be dropped.

3

Define the *vertex boundary* $B(\mathcal{A})$ of a subset $\mathcal{A} \subset \mathcal{P}[n]$.

State and prove Harper's inequality for the minimum size of the vertex boundary. [You may assume, if you wish to, the Kruskal–Katona theorem.]

For $i, j \in [n]$ let C_{ij} be the usual compression operator. Is it true that, if $\mathcal{A} \subset [n]^{(r)}$, then $|B(C_{ij}\mathcal{A})| \leqslant |B(\mathcal{A})|$?

4

Let $n \equiv 0 \pmod 4$ and let $\mathcal{A} \subset \mathcal{P}[n]$ satisfy $|A \cap B| \neq n/4$ for all $A, B \in \mathcal{A}$. Prove that $|\mathcal{A}| \leqslant 1{\cdot}999^n$. [You may assume that $\sum_{j \leqslant \gamma n} \binom{n}{j} \leqslant 2^{H(\gamma)n}$ for $0 \leqslant \gamma \leqslant 1/2$. You may also assume Harper's inequality.]

Suppose now that $\mathcal{A}$ satisfies $|A \cap B| \neq \ell$ for all $A, B \in \mathcal{A}$. Indicate how to modify your proof to show $|\mathcal{A}| \leqslant 1{\cdot}999^n$ if $\ell = \lfloor n/8 \rfloor$.

If $\ell = \lfloor 2n/3 \rfloor$, must $|\mathcal{A}| \leqslant 1{\cdot}999^n$ still hold?

END OF PAPER