UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2012 9:00 am to 11:00 am

PAPER 10

RAMSEY THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Hales–Jewett theorem, and deduce van der Waerden's theorem. Prove the strengthened van der Waerden theorem.

Is the following strengthening of the Hales–Jewett theorem true or false: for any m and k there exist n and d such that whenever $[m]^n$ is k-coloured there exists a monochromatic line whose active coordinate set has size d?

2

Show that if $a_1, \ldots, a_n$ are non-zero rationals then the matrix $(a_1, \ldots, a_n)$ is partition regular if and only if some (non-empty) subset of the a_i has sum zero.

[You may assume van der Waerden's theorem. No form of Rado's theorem may be assumed without proof.]

For an $m \times n$ matrix A, and a subset S of the positive integers, we say that A is *partition regular over* S if whenever S is finitely coloured there exists a vector $x \in S^n$, with all its entries having the same colour, such that $Ax = 0$.

(i) Show that if $a_1, \ldots, a_n$ are non-zero rationals then the matrix $(a_1, \ldots, a_n)$ is partition regular over the set of even positive integers if and only if it is partition regular.

(ii) Show that if $a_1, \ldots, a_n$ are non-zero rationals then the matrix $(a_1, \ldots, a_n)$ is partition regular over the set of odd positive integers if and only if the sum of all the a_i is zero.

3

Prove that there exists an idempotent ultrafilter in $\beta\mathbb{N}$.

[You may assume that $\beta\mathbb{N}$ is a (non-empty) compact Hausdorff space, and that the operation $+$ on $\beta\mathbb{N}$ is associative and left-continuous.]

Deduce Hindman's theorem.

[You may assume simple properties of ultrafilters and their quantifiers.]

(i) Does there exist an idempotent ultrafilter that has as a member the set $\{x \in \mathbb{N} : x \text{ is not a multiple of } 10\}$?

(ii) Does there exist an idempotent ultrafilter that has as a member the set $\{x \in \mathbb{N} : x \text{ is not a power of } 10\}$?

4

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *completely Ramsey*? Define the $*$-topology on $\mathbb{N}^{(\omega)}$, and explain what it means for a subset of $\mathbb{N}^{(\omega)}$ to be $*$-*Baire*.

Give an example of a set that is not Ramsey, and also an example of a set that is Ramsey but not completely Ramsey.

Prove that a subset of $\mathbb{N}^{(\omega)}$ is completely Ramsey if and only if it is $*$-Baire.

[You may assume that every $*$-open set is completely Ramsey.]

Is $\mathbb{N}^{(\omega)}$ $*$-meagre? Is it τ-meagre? Justify your answers.

END OF PAPER