UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 1

LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let L be a finite dimensional Lie algebra.

Define the adjoint map $ad_x : L \longrightarrow L$ for x in L.

Define what is meant by L being nilpotent.

Show that L is nilpotent if and only if ad_x is nilpotent for all x in L.

2

Define what is meant by a Lie algebra L being soluble.

Show that all two dimensional complex Lie algebras are soluble.

Prove that all irreducible representations of a finite dimensional complex soluble Lie algebra are one dimensional.

3

Define what is meant by a finite dimensional complex Lie algebra L being semisimple.

Define the Killing form B_L and show that it is non-degenerate.

Define what is meant by a Cartan subalgebra H of L.

Show that the restriction of B_L to any Cartan subalgebra H of L is also non-degenerate.

4

Define what is meant by a representation of a Lie algebra being completely reducible.

Show that all finite dimensional representations of a semisimple finite dimensional complex Lie algebra are completely reducible.

5

State the classification of weighted Dynkin diagrams of irreducible root systems.

Define what is meant by a base Δ of a root system Φ and define the Cartan matrix of Φ with respect to Δ.

Define the Weyl group of Φ.

For the root system of type C_3 construct the corresponding semisimple complex Lie algebra. Describe its Weyl group. Sketch the root system showing the Weyl chambers.

6

Let L be a semisimple finite dimensional complex Lie algebra with Cartan subalgebra H.

Define what is meant by a primtive element of weight ω where ω lies in H^*.

Show that for each ω in H^* there is an irreducible representation with a primitive element of weight ω.

Define the fundamental weights of L. Describe them in the case of sl_3.

END OF PAPER