UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 9:00 am to 12:00 pm

PAPER 9

AN INTRODUCTION TO PDE'S IN KINETIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight and each question accounts for **40%** of the marks.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question is concerned with various properties of the linear transport equation.

(a) State the Newton laws for N particles with interaction binary potential and some external potential.

(b) Write the Hamiltonian form of the previous equations.

(c) Define the associated transport equation and its characteristics curves (still in Hamiltonian form). We assume that all these equations have smooth unique global solutions for given smooth initial data.

(d) Prove that (using the general Hamiltonian form) the application $S_t : (x, v) \to (X(t, x, v), V(t, x, v))$ (where $X(t, x, v)$ and $V(t, x, v)$ are solutions at time t of the ODE system starting from x, v) has Jacobian equal to 1 for all times.

(e) Give two proofs of the conservation of L^p norms along time for the solutions to the transport equation, for $p \in [1, +\infty)$ (one proof should use characteristics and the previous item (d), the other one should not use characteristics).

(f) Consider the two-dimensional vortices incompressible Euler equation:

$$\partial_t \omega + u \cdot \nabla_x \omega = 0, \quad \omega = \omega(t, x) \in \mathbb{R}, \ x \in \mathbb{R}^2, \quad u := \nabla_x^{\perp} \Delta_x^{-1} \omega$$

with some initial conditions $\omega_{in} \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ and where

$$\nabla_x^{\perp} \phi = (-\partial_{x_2} \phi, \partial_{x_1} \phi).$$

Prove formally (i.e. assuming existence of smooth solutions and that all integration by parts are justified) the conservation of L^p norms along time for $p \in [1, +\infty)$.

(g) We now want to prove some *growth* criterion on the force field in order to ensure global existence for the transport equation. Let us start with an ODE result. Consider $F \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ ($d \in \mathbb{N}$) and the ODE

$$y'(t) = F(t, y), \quad t \geqslant 0, \quad y \in \mathbb{R}^d$$

with initial condition $y(0) = y_0$ at time $t = 0$. Consider the maximal time of existence $T_c \in (0, +\infty]$ given by the Picard-Lindelöf theorem. Prove that if

$$\sup_{t \to T_c^-} |y(t)| < +\infty$$

then $T_c = +\infty$.

(h) Consider a function $F \in C^1(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that

$$\forall\, x, v \in \mathbb{R}^d, \quad |F(t, x, v)| \leqslant C(1 + |x| + |v|)$$

for some constant $C \in (0, +\infty)$. Prove that the transport equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \quad f = f(t, x, v), \ x \in \mathbb{R}, \ v \in \mathbb{R}^d$$

with initial data $f_{in} \in C^1(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ has a unique global solution in C^1.

(i) Show by producing a counter-example that the conditions assumed on F in the previous item do **not** imply that F is uniformly Lipschitz with respect to (x, v) on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

2

We have seen during the lectures how to construct a modified norm of hypocoercivity for the linear relaxation equation. Let us look at another method of building such a modified norm of hypocoercivity by using general semigroup arguments, and discuss its limits.

(a) Recall what is the linear relaxation equation and state the result of hypocoercivity in the torus for this equation that we have seen in the lecture (you do not need to state the exact formula for the modified norm).

(b) Prove this result in the case of *homogeneous* solutions.

(c) Consider a bounded operator L in a Hilbert space H. (The operator defining our equation is unbounded, but for simplicity we shall perform the semigroup argument in this simpler setting. It can easily be extended to the linear relaxation equation.)

We shall prove a preliminary result. Assume that the semigroup T_t of L satisfies

$$\forall f \in H, \quad \|T_t f\|_H \leqslant C\, e^{\lambda t} \|f\|_H \tag{1}$$

for some $C \in [1, +\infty)$ and $\lambda \in \mathbb{R}$. Prove that

$$(A) \quad \text{one can choose } C = 1 \text{ in (1)}$$

if and only if L satisfies

$$(B) \quad \forall f \in H, \quad \langle Lf, f \rangle_H \leqslant \lambda \|f\|_H^2.$$

A *hypocoercive* situation is when the semigroup satisfies (1) in the ambient norm, but only with some $C > 1$.

(d) Assume (1) with $C \in (1, +\infty)$ and $\lambda < 0$ and consider

$$\|f\|_* := \left(\eta \|f\|^2 + \int_0^{+\infty} \|T_\tau f\|_H^2 \, d\tau \right)^{1/2}, \quad \eta > 0.$$

Prove that this is well-defined as a Hilbert norm, and that, for any $\eta > 0$ this norm is equivalent to the original norm $\| \cdot \|_H$.

(e) Under the same assumption and for $\eta \in (0, \eta_0)$ where η_0 should be computed, prove that S_t satisfies, **in the modified norm** $\| \cdot \|_*$, the control (1) with $C' = 1$ and some λ' to be computed.

(f) Discuss the limits of this general argument as compared to the results and methods of hypocoercivity we have seen during the lectures (compare assumptions, conclusions, and the modified norm itself).

3

This exercise is concerned with the so-called *Fokker-Planck operator*. Consider $\Phi \in C^2(\mathbb{R}^d, \mathbb{R})$ and consider the associated operator

$$Lf := \nabla_v \cdot (\nabla_v f + \Phi(v) f), \quad f = f(t,v), \ v \in \mathbb{R}^d.$$

We assume that the potential Φ satisfies

$$\int_{\mathbb{R}^d} e^{-\Phi(v)} dv = 1$$

as well as

$$\lim_{|v| \to +\infty} \left(\frac{|\nabla \Phi(v)|^2}{2} - \Delta \Phi(v) \right) = +\infty.$$

(a) Compute the functions $f \in C^2(\mathbb{R}^d; \mathbb{R}) \cap W^{1,1}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |f| \, |\nabla \Phi| \, dv < +\infty$$

and which cancel the operator L, i.e. $Lf = 0$.

(b) From now on we assume that f is always sufficiently smooth and sufficiently decaying at infinity so that integrals and integration by parts all make sense. Prove that

$$I(f) := -\langle Lf, f \rangle_{L^2(M^{-1})} \geqslant 0$$

where $L^2(M^{-1})$ is the Lebesgue L^2 space on $\mathbb{R}^d$ with reference measure M^{-1} with $M(v) := e^{-\Phi(v)}$.

(c) We shall prove that this operator has a *spectral gap* in the space $L^2(M^{-1})$. In other words we shall prove a *Poincaré inequality* for the measure M on $\mathbb{R}^d$. Prove that

$$I(f) \geqslant \frac{1}{2} \int_{\mathbb{R}^d} f^2 \left(\frac{|\nabla \Phi(v)|^2}{2} - \Delta \Phi(v) \right) M^{-1} dv \geqslant \lambda_1 \int_{|v| \geqslant R} f^2 M^{-1} dv \qquad (1)$$

for some constant $\lambda_1 \in (0, +\infty)$ independent of f.

Hint: Plug $f = \sqrt{M} g$ in the formula for $I(f)$.

(d) You can admit the following particular case of the *Rellich-Kondrashov Theorem*: the embedding $H^1(B(0,R)) \to L^2(B(0,R))$ is compact for any $R > 0$ (meaning that it maps bounded sets into relatively compact sets), where $B(0,R)$ denotes the usual euclidean ball with radius R. Prove that

$$\int_{B(0,R)} \left| \nabla \left(\frac{f}{M} \right) \right|^2 M dv \geqslant C \int_{B(0,R)} \left[f - \left(\int_{B(0,R)} f(v_*) dv_* \right) M(v) \right]^2 M^{-1} dv$$

for some constant $C \in (0, +\infty)$ independent of f.

Hint: Argue by contradiction and suppose that there a sequence $h_n = f_n / M$ such that

$$\forall n \geqslant 1, \quad \int_{B(0,R)} |\nabla h_n|^2 M dv \leqslant \frac{1}{n}$$

and

$$\forall\, n \geqslant 1, \quad \int_{B(0,R)} \left[h_n - \left(\int h_n M \right) \right]^2 M dv = 1.$$

(e) Deduce that

$$I(f) \geqslant \lambda_2 \int_{B(0,R)} \left[f - \left(\int_{\mathbb{R}^d} f(v_*) dv_* \right) M(v) \right]^2 M^{-1} dv - \lambda_3 \int_{|v| \geqslant R} f^2 M^{-1} dv \quad (2)$$

for some constants $\lambda_2, \lambda_3 \in (0, +\infty)$ independent of f.

(f) By combining (1) and (2), conclude that

$$I(f) \geqslant \lambda_4 \int_{\mathbb{R}^d} \left[f(v) - \left(\int_{\mathbb{R}^d} f(v_*) dv_* \right) M(v) \right]^2 M^{-1} dv$$

for some constant $\lambda_4 \in (0, +\infty)$ independent of f.

4

Write an essay on the Vlasov-Poisson equation, including a presentation and as many properties, remarks and proofs as possible.

END OF PAPER