UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 9:00 am to 12:00 pm

PAPER 8

ANALYTIC TOPICS IN GROUP THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What does it mean to say that a group has polynomial growth of exponent at most d? Show that if G is virtually $\mathbb{Z}^k$ for some $k \in \{0, 1, 2, 3\}$ then G has cubic growth, that is to say polynomial growth of exponent at most 3.

Show that if G has cubic growth then G is virtually $\mathbb{Z}^k$ for some $k \in \{0, 1, 2, 3\}$. [*Any result proved in lectures may be used, provided it is clearly stated. Basic results about nilpotent groups may also be used without detailed comment.*]

2

Show that SO(3) contains a pair of elements a, b with the property that $w(a, b)$ has distance at least $C^{-\ell(w)}$ from the identity for any nontrivial word w, where $\ell(w)$ denotes the length of the word w and C is an absolute constant. [*The distance on SO*(3) *is the one induced from the Euclidean distance on* $\mathbb{R}^9$.]

What is an amenable group? Prove that SO(3) is not amenable. [*You may assume the axiom of choice.*]

3

Outline a proof of the following fact: a finitely-generated group G with polynomial growth admits a finite-dimensional unitary representation $\rho : G \to \mathrm{U}_n(\mathbb{C})$ with infinite image. Include at least some of the more interesting technical details.

4

Consider a random walk on a finitely-generated group G, starting at the identity and in which the steps are drawn uniformly at random from a symmetric set S. Show that the probability of being at the identity at time n tends to zero as $n \to \infty$ if and only if G is infinite.

Now suppose that G has a finite-index subgroup isomorphic to $\mathbb{Z}^2$. Show that the random walk is recurrent. [*Basic results on growth in groups need not be proven if stated correctly.*]

END OF PAPER