UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 1:30 pm to 3:30 pm

# PAPER 75

# NON-NEWTONIAN FLUID DYNAMICS

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Consider an Oldroyd-B fluid with density $\rho$, viscosity $\mu$, elastic modulus $G$ and relaxation time $\tau$, so that

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E} + G\mathbf{A},$$
$$\frac{D\mathbf{A}}{Dt} - (\boldsymbol{\nabla}\mathbf{u})^T \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u} + \frac{1}{\tau}(\mathbf{A} - \mathbf{I}) = 0.$$

Show that in a time-dependent but uniform shear flow $u = (\gamma(t)y, 0, 0)$ the shear stress is given by

$$\sigma_{12}(t) = \mu\gamma(t) + G\int^t \gamma(t')e^{-(t-t')/\tau}\, dt'.$$

Now consider the same fluid occupying the half-space above a plate at $y = 0$ which is oscillating in its own plane with velocity $(\mathrm{Re}(U_0 e^{i\omega t}), 0, 0)$. Taking the flow to be of the form $u(y,t) = (\mathrm{Re}(U_0 e^{ky+i\omega t}), 0, 0)$, find an expression for $k^2$. In the limit $1/\tau \ll \omega \ll G/\mu$, show that there is a decaying elastic wave, with wavelength $2\pi c/\omega$ where $c^2 = G/\rho$, and with a rate of exponential decay in space

$$\frac{\omega}{2c}\left(\frac{1}{\omega\tau} + \frac{\mu\omega}{G}\right).$$

Under what conditions is the Newtonian limit recovered.

**2**

Consider a Johnson-Segalman fluid in the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E} + G\mathbf{A},$$
$$\frac{D\mathbf{A}}{Dt} + \boldsymbol{\Omega} \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\Omega} - \alpha(\mathbf{E} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{E}) + \frac{1}{\tau}(\mathbf{A} - \mathbf{I}) = 0,$$

where $\mathbf{E}$ is the symmetric and $\boldsymbol{\Omega}$ the antisymmetric part of $\boldsymbol{\nabla}\mathbf{u}$, $\mu$, $G$ and $\tau$ are positive constants and $|\alpha| < 1$.

In steady simple shear $u = (\gamma y, 0, 0)$, show that

$$A_{13} = A_{31} = A_{23} = A_{32} = 0, \quad A_{33} = 1,$$
$$A_{11} = 1 + (1+\alpha)\gamma\tau A_{12}, \quad A_{22} = 1 - (1-\alpha)\gamma\tau A_{12},$$

and find $A_{12}$.

Sketch the viscosity and normal stress differences as functions of the shear rate.

Show that the shear stress $\sigma_{12}$ is monotonic in the shear rate $\gamma$ only if $\mu > \alpha G\tau/8$.

[*Hint: find the maximum negative slope of* $x/(1+x^2)$.]

**3**

A Bingham fluid has yield stress $\sigma_Y$ and viscosity $\mu$ so that in a simple shear flow with shear rate $\gamma$

$$\begin{cases} \gamma = 0 & \text{if} \quad |\sigma| \leqslant \sigma_Y, \\ \sigma = \operatorname{sign}(\gamma)\sigma_Y + \mu\gamma & \text{if} \quad |\sigma| > \sigma_Y. \end{cases}$$

A pressure difference $\Delta p$ is applied to a long tube of length $L$ and radius $a$ filled with the above Bingham fluid. Use a force balance to explain why the axial component of the steady momentum equation takes the form

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right).$$

Find the volumetric flow rate. Show that

$$Q = 0 \quad \text{if} \quad \frac{\Delta p}{L} < \frac{2\sigma_Y}{a},$$

$$\text{and} \quad Q \sim \frac{\pi a^4}{8\mu}\left(\frac{\Delta p}{L} - \frac{8\sigma_Y}{3a}\right) \quad \text{if} \quad \frac{\Delta p}{L} \gg \frac{\sigma_Y}{a}.$$

Now suppose that the tube is held vertically and allowed to drain freely ($\Delta p = 0$) under gravity. Discuss, without detailed calculation, what might be seen after a long time.

## END OF PAPER