UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 9:00 am to 12:00 pm

PAPER 73

SOLIDIFICATION OF FLUIDS

A distinction can be gained from substantially complete answers to ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

This is an ***OPEN BOOK*** *examination.*
Candidates may bring any course handouts (including example sheets) and any handwritten material into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A deep layer of lava at its freezing temperature, T_m, loses heat from its upper surface to air of fixed temperature T_∞. The radiative flux of heat from the upper surface is approximately given by

$$F_s = \rho c_p \lambda_R (T_s - T_\infty),$$

where ρ is the density of the lava, c_p is its specific heat capacity, λ_R is a constant, $T_s(t)$ is the surface temperature of the lava and t is the time since formation of a solid crust of thickness $a(t)$. Assume that the thermal field is approximately linear throughout the crust, and find an algebraic equation for $a(t)$. How does a vary with t at small and large times?

For very deep lava intrusions the pressure-dependence of the bulk freezing temperature can lead to further solidification at depth. Consider a pressure (and therefore depth) dependent freezing temperature of the form

$$T_e(z) = T_m + \Delta T_m z/H,$$

where T_0 and ΔT_m are constants, z is the vertical depth from the surface, and H is the initial depth of the intrusion. Assume that the liquid part of the intrusion is well mixed (of uniform temperature $T_e(b)$) so that a solid basal layer (also of uniform temperature $T_e(b)$) forms of thickness $H - b(t)$.

For a well-mixed fluid interior the heat flux from the liquid to the crust is given by

$$F_c = \rho c_p \lambda_c (T_e - T_m),$$

where $\lambda_c < \lambda_R$ is a constant.

Use conservation of heat (or otherwise) to derive evolution equations for the thickness of the basal layer and the crust in the limit where the crust remains thin, $a \ll H$. Solve for the thickness of the lower layer explicitly, and provide estimates for the behaviour of $a(t)$ at small and large times.

2

Sunlight of intensity I_0 is incident normally on a block of ice of far-field temperature T_m, which then sublimates (vaporizes) at its surface. You may assume that the surface temperature is therefore also equal to T_m. The light energy absorbed per unit volume at distance z below the surface of the ice is

$$Q = \lambda I_0 e^{-\lambda z},$$

where λ is a constant. Find the temperature field $T(z)$ in the ice and the rate of sublimation V. Draw a sketch of $T(z)$ in the case $\lambda \gg V/\kappa$, where κ is the thermal diffusivity of ice.

Examine the morphological stability of the surface of the ice, ignoring any heat transfer to the air but taking account of the Gibbs-Thomson relationship

$$T = T_m - \gamma \nabla \cdot \mathbf{n},$$

where $\mathbf{n}$ is the unit normal to the solid–vapour interface pointing into the vapour, and γ is a constant that characterises the surface energy. In particular, you should find the growth rate σ as a function of the wavenumber α of transverse disturbances to the surface of the ice.

By considering the conditions for marginal equilibrium ($\sigma = 0$) or otherwise, show that in the limit $\lambda \gg V/\kappa$ the interface is morphologically unstable if

$$I_0 > \frac{27}{4} k \gamma \lambda^2,$$

where k is the thermal conductivity of ice.

[*Hint: assume that the critical condition occurs when $\alpha = O(\lambda)$, so that $\alpha \gg V/\kappa$ and justify this assumption from your answer.*]

3

At sufficiently large undercooling nucleation of crystals can occur predominantly within the bulk liquid, away from cooled boundaries. Consider a turbulent plume of water and suspended ice crystals rising vertically along a cooled boundary of temperature $T_b < T_m$, where T_m is the bulk melting temperature. The buoyancy of the plume derives principally from the presence of the ice crystals of density ρ_i, which is less than the density of water ρ_w, which can be assumed constant, independent of temperature. You may assume that crystal growth is sufficiently rapid to maintain thermodynamic equilibrium in the plume, which is cooled by the wall and entrains water of far-field temperature $T_\infty > T_m$.

Use an appropriate control-volume approach to derive equations for conservation of mass and momentum in steady flow, including the drag exerted by the vertical wall and accounting for the different densities of ice and water. You may assume that the drag coefficient is independent of the volume fraction ϕ of ice crystals.

Now assuming that the densities and thermal properties are equal between phases, show that the expression for conservation of heat in the plume is given by

$$\frac{d}{dz}\left[bw\left\{c_p(T_m - T_\infty) - L\phi\right\}\right] = c_p \; St \; w \; (T_b - T_m),$$

where z is the vertical coordinate, b the width of the plume, w the mean vertical velocity within the plume, c_p the specific heat capacity, L the latent heat and St the thermal Stanton number.

Find a similarity solution for the rise of the plume, ignoring the density variations associated with changes in ϕ except where they modify the buoyancy.

What is the ice production rate as a function of height within the plume? At what temperature must the vertical wall be held to maintain the plume?

4

A porous medium of uniform permeability Π_l is saturated with an aqueous salt solution and is pulled at constant speed V to colder temperatures through a fixed temperature gradient G. The solution partially solidifies to form a mushy region of uniform permeability Π_m in $-a < z < 0$ and a eutectic solid in $z < -a$. The solution is introduced at $z = h$ with concentration C_0 and vertical component of the Darcy velocity $w = -U$ in a Cartesian coordinate system (x, z).

Assume that the system has infinite extent in the x direction and determine the forced Darcy flow (u, w) with $u = x\, f'(z)$, where $f(z)$ is to be determined in each medium. You may neglect the influence of gravity. In particular, show that the vertical velocity in the unfrozen porous region $0 < z < h$ is

$$w = -U \frac{z + a\,(\Pi_m/\Pi_l)}{h + a\,(\Pi_m/\Pi_l)}$$

and write down an expression for the vertical component of velocity in the mushy region.

Consider the combined limits $a\Pi_m/h\Pi_l \ll 1$, $h \gg D/V$, where D is the diffusivity of salt in solution, and solve for the concentration field $C(z)$ in the unfrozen porous region using the marginal equilibrium condition at the mush–liquid interface, given the liquidus

$$T_L(C) = -mC,$$

where m is constant, and the frozen temperature field

$$T = T_E + G(z + a),$$

where T_E is the eutectic temperature. Use this solution to show that the thickness of the mushy region is

$$a = \frac{-mC_0 - T_E}{G} - \left(\frac{\pi h D}{2U}\right)^{1/2} \exp\left(\frac{V^2 h}{2UD}\right) \operatorname{erfc}\sqrt{\frac{V^2 h}{2UD}}.$$

If $V^2 h/2UD \gg 1$ and $D/V \ll h$ then $C(0) \approx C_0$. With this limit in addition to the limits above, show that the bulk composition

$$(1 - \phi)C = C_0 + \frac{\Pi_m}{\Pi_l}\frac{U}{V}\frac{G}{m}\frac{(z + a)^2 - a^2}{2h},$$

where ϕ is the solid fraction of ice, and sketch the trajectory of (C, T) in the phase diagram. What is the composition of the composite eutectic solid in $z < -a$?

END OF PAPER