UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 1:30 pm to 4:30 pm

PAPER 72

GEOPHYSICAL AND ENVIRONMENTAL FLUID DYNAMICS

*You may attempt **ALL** questions, although full marks can be achieved by good answers to **THREE** questions. Completed answers are preferred to fragments.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a channel with a triangular cross-section with boundaries at $y = \pm\frac{1}{2}b(x,z)$ where

$$b(x,z) = \begin{cases} B(x)(z - H(x)) & z \geqslant H(x) \\ 0 & \text{otherwise.} \end{cases}$$

Here, x is in the along-channel direction, z is vertically upwards, $z = H(x)$ is the elevation of the deepest part of the channel and $B(x)$ describes the overall variation in the channel width. The channel contains water with a free surface at $z = H(x) + h(x,t)$ beneath an infinite atmosphere. Suppose it is raining so that water is being added to the channel at a volume flux of $R(x)$ per unit length of the channel independently of the channel width.

(a) What conditions must be satisfied to be able to describe the flow using the inviscid shallow water approximation? Assuming this approximation and that the rain water has no net horizontal momentum when it joins the channel, derive the appropriate continuity equation and show that the momentum equation can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial}{\partial x}(H + h) = -2\frac{Ru}{Bh^2}.$$

State any additional assumptions that are necessary.

(b) Show that this system is hyperbolic. Determine the characteristics λ and the ordinary differential equations describing the evolution of the flow along these characteristics. Write your answer in terms of the long wave speed c rather than depth h.

(c) Define the Froude number F and state what is meant by the terms ‘subcritical’, ‘critical’ and ‘supercritical’ in the context of a hydraulic flow. Assume that we have a steady flow along the channel with the volume flux at $x = 0$ given by Q_0. Give an expression for the volume flux $Q(x)$. Determine how the Bernoulli potential $J(x)$ changes along the channel due to the rainfall, assuming the rain is well mixed throughout the cross-section but in the absence of any hydraulic jumps or other dissipative processes.

(d) For uniform rain $R(x) = R_0$ in a channel with $B(x) = 1$ we can model the local change in Bernoulli potential as $J(x) = J_0 - \beta x$ near $x = 0$ for some $J_0 > 0$ and $\beta > 0$. Define the specific energy E and determine the slope dH/dx required so that a flow of depth h_0 at $x = 0$ is critical at $x = 0$. Express your answer in terms of h_0, R_0 and β.

2

Following an unusual weather event the surface of the ocean has an elevation at $t = 0$ given by

$$z = \begin{cases} 0 & x < -L \\ \eta_0 & -L \leqslant x \leqslant L \\ 0 & x > L \end{cases}$$

The ocean can be treated as an inviscid shallow water flow of homogeneous density with its bottom located at $z = -H + \beta y$. The ocean is at rest at $t = 0$ relative to the rotating frame of reference described by constant Coriolis parameter f.

(a) Define a Rossby number Ro appropriate for this ocean. Write down continuity and momentum equations, treating the ocean as a shallow water flow of local depth h and velocity (u, v, w). Prove that the potential vorticity,

$$\Pi = \frac{\zeta + f}{h},$$

is conserved by a fluid parcel. Here $\zeta = \partial v/\partial x - \partial u/\partial y$ is the relative vertical vorticity.

(b) Simplify the equations of motion in the limit $Ro \ll 1$ and linearise Π under the assumptions $\eta_0 \ll H$ and $\beta = 0$. By writing the local depth as $h = H + \eta$, decompose the horizontal velocity and depth perturbation as

$$(u, v, \eta) = (u_\infty, v_\infty, \eta_\infty) + (u', v', \eta').$$

Determine the steady component of the flow, $(u_\infty, v_\infty, \eta_\infty)$, as $t \to \infty$. What balance does the Rossby radius of deformation, $R = (gH)^{1/2}/f$, represent? Sketch the resulting steady velocity and depth profiles. Derive also the dispersion relation for the transient flow (u', v', η'). Determine the corresponding phase and group velocities as a function of the wavenumber vector $\mathbf{k}$. You need not solve the transient problem for the initial conditions.

(c) Explain why Rossby waves will be excited if $\beta \neq 0$. Determine the dispersion relation, phase and group velocities for these waves under the assumption that $\eta_0 \ll H$ and that β is small. Prove that the initial adjustment of the surface occurs much faster than the Rossby waves provided $\beta R \ll H$.

3

Consider a body of fluid of density ρ_0 containing suspended particles of density ρ_p and volume concentration ϕ. The settling speed of the particles in a quiescent fluid is given by $W = W_0(1-\phi)^\alpha$ for some constant α, where W_0 is the settling speed in a dilute suspension.

(a) Explain why the particle sedimentation rate decreases as the particle concentration increases. Derive a one-dimensional equation describing the evolution of the particle concentration in a quiescent fluid. Determine the characteristics λ for this equation. What quantity is conserved along the characteristics? For the case $\alpha = 1$, describe the evolution of the concentration field in a container of depth H where the initial density concentration is given by $\phi(z, t = 0) = (1 - z/H)\phi_0$ for constant $\phi_0 < 1$. Show that a concentration shock will form within the container if $\phi_0 > \frac{1}{2}$ and determine the time and height at which the shock will form.

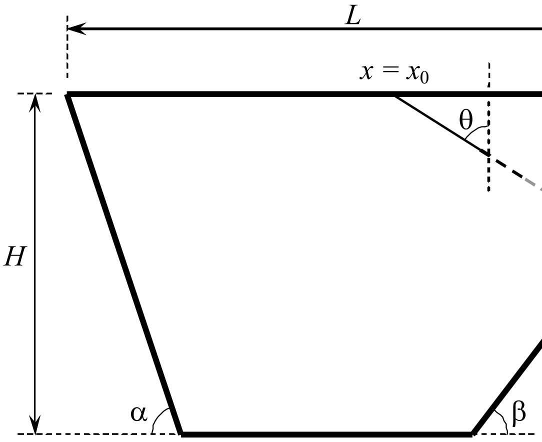
(b) Suppose there is a source of energy keeping the contents of the container well mixed with uniform concentration $\phi(t)$. Explain why particles that come close to the lower boundary can still settle out of the flow. Derive an equation for $\phi(t)$ and solve this for $\alpha = 1$ given that $\phi(t = 0) = \phi_0$.

(c) Consider a lock-release particle-laden gravity current in a channel of unit width. The fluid of density ρ_0 is of infinite depth above the channel's flat base, but the fluid behind the lock of length L_0 contains suspended particles with a uniform concentration ϕ_0 to a depth H. Derive an integral (box) model for the evolution of the current. State any assumptions made. Solve this model for the run-out length of the current when $\alpha = 1$ under the assumption that the flow is turbulent and the volume of the current is constant.

(d) Repeat the calculation of part (c) but in an axisymmetric geometry with the particle-laden fluid initially confined to $0 \leqslant r \leqslant R$ and $0 \leqslant z \leqslant H$.

4

Inertial-gravity waves of frequency ω are excited in a density stratified basin of depth H rotating with an angular velocity $f/2$. The basin is square at the surface with sides of length L and the stratification of the fluid it contains is characterised by a constant buoyancy frequency N. The front and back boundaries are vertical. However, the left-hand boundary makes an angle α to the horizontal and the right-hand boundary makes an angle β to the horizontal, as shown in the figure below.



(a) By linearising the equations of motion, show that the vertical velocity of the inertial-gravity waves is governed by

$$\left[\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + f^2\frac{\partial^2}{\partial z^2}\right] w = 0$$

for a Boussinesq system. Determine the dispersion relation for these waves and demonstrate that the phase and group velocities are perpendicular.

(b) Outline the idea behind the formation of an inertial-gravity wave attractor for plane waves with the wavenumber vector confined to a plane parallel to the front and back boundaries of the basin. Consider a parcel of energy propagating at an angle θ to the vertical. Determine the location $x = x_4$ at which this parcel will reach the top boundary if it starts propagating down and to the right from a position $x = x_0$ on the top boundary and reflects from each of the other boundaries exactly once. Determine the location and focusing power of the corresponding wave attractor. What direction does the energy on the attractor move around the domain? What happens if $\alpha = \beta$?

(c) For $\alpha > \beta$, over what range of θ can this simple four-reflection attractor exist? What restriction must be placed on the aspect ratio H/L?

END OF PAPER