UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Tuesday, 7 June, 2011   1:30 pm to 4:30 pm

PAPER 71

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For each fixed real α and $\epsilon \to 0$, find two terms of an asymptotic expansion for each eigenvalue of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ \epsilon & \alpha & 1 \\ 0 & \epsilon & 2 \end{pmatrix} .$$

[Hint: for some values of α the standard expansion fails. These values should be identified and an alternate expansion should be found.]

Consider the distinguished limit in which both $\alpha \to 1$ and $\epsilon \to 0$. Find a suitable asymptotic expansion for the eigenvalues in this limit and verify that it agrees in an appropriate sense with your earlier results.

(b) Obtain two terms of an asymptotic expansion for $\tan^{-1} x$ as $x \to \infty$.

In the limit $\epsilon \to 0$ find two terms of an asymptotic expansion for

$$f(\epsilon) = \int_0^{\pi/2} \frac{\cos^2 x}{x^2 + \epsilon^2} \mathrm{d}x \,.$$

[The second term may be left as a definite integral that does not involve ϵ.]

2

(a) Explain *briefly* what is meant by the *method of multiple scales*, indicate the class of problems for which it might be used, and illustrate it by means of the problem below.

The amplitude $\theta(t)$ of a pendulum satisfies the equation

$$\ddot{\theta} + \sin\theta = 0 \quad \text{with } \theta = \epsilon \text{ and } \dot{\theta} = 0 \text{ at } t = 0.$$

For small amplitudes of swing ($\epsilon \ll 1$) use multiple scales to find $\theta(t)$ and hence determine the period of oscillation correct to the first term involving ϵ.

(b) Suppose that

$$f(x) = \int_a^b g(t) e^{x\phi(t)} \mathrm{d}t$$

and that $\phi(t)$ has a maximum at $t = c$ with $a < c < b$. Obtain *two* terms of an asymptotic expansion for f as $x \to \infty$ making clear any assumptions you make.

$$\left[\int_{-\infty}^{\infty} e^{-s^2/2} s^{2n} \mathrm{d}s = \sqrt{2\pi}(2n-1)(2n-3)...3.1 \right]$$

3

(a) The function $y(x;\epsilon)$ satisfies

$$(x+\epsilon y)y' + y = 1 \qquad \text{with } y(1) = 2$$

and $\epsilon \ll 1$. By considering an inner region of size $O(\epsilon^{1/2})$ around $x = 0$, show that the first two terms in $y(0)$ are

$$y(0) = \sqrt{\frac{2}{\epsilon}} - \frac{1}{\sqrt{2}}\ .$$

(b) Consider the first-order ordinary differential equation

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{q}\ ,$$

where $\boldsymbol{q}$ is a two-dimensional vector and $\boldsymbol{A}$ is a 2×2 constant real matrix which has eigenvalues λ_1, λ_2 with $\mathrm{Re}(\lambda_1) > \mathrm{Re}(\lambda_2)$. The optimal growth $G(t)$ is defined to be the maximum value of

$$\frac{\|\boldsymbol{q}(t)\|}{\|\boldsymbol{q}(0)\|}$$

over all $\boldsymbol{q}(0) \neq 0$. Show that

$$G(t) \leqslant \kappa e^{\mathrm{Re}(\lambda_1)\mathrm{t}}\ ,$$

where $\kappa = \|\boldsymbol{F}\|\ \|\boldsymbol{F}^{-1}\|$ and $\boldsymbol{F}$ possesses as its columns the normalised eigenvectors of $\boldsymbol{A}$. Calculate the upper bound on $G(t)$ explicitly for

$$\boldsymbol{A} = \begin{pmatrix} 1 & -\alpha \\ \alpha & -1 \end{pmatrix}\ ,$$

where $0 \leqslant \alpha \leqslant 1$.

4

Describe the use of the Briggs-Bers method to determine the long-time behaviour of the solutions of the linearised Ginzburg Landau equation

$$\frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} - \gamma\frac{\partial^2 A}{\partial x^2} - \mu A = 0$$

in $-\infty < x < \infty$, where U, μ, γ are real constants with $\gamma > 0$.

Now consider the linearised Ginzburg Landau equation in $x \geqslant 0$ subject to $A(0,t) = 0$ and $A(x,t) \to 0$ as $x \to \infty$, and suppose that μ is no longer constant but takes the form $\mu = \mu_0 - \lambda\epsilon x$, with $\epsilon \ll 1$ and μ_0 and λ positive real constants.

By considering a solution of the form

$$A(x,t) = f(\epsilon^\sigma x)\exp\left(\frac{Ux}{2\gamma} - \mathrm{i}\omega t\right) ,$$

where the index $\sigma > 0$ and the function f are to be determined, show that the global mode frequencies are

$$\mathrm{i}\left(\mu_0 - \frac{U^2}{4\gamma} + (\epsilon^2\gamma\lambda^2)^{1/3} z_n\right) \quad \text{for } n = 1, 2, \ldots ,$$

where z_n is the nth zero of the Airy function $\mathrm{Ai}(x)$ on the negative real axis. Hence, deduce the condition for global stability. How does this compare to the condition for local absolute instability?

[Hint: $\mathrm{Ai}(x)$ is the solution of $y'' = xy$ which tends to zero like $\exp(-2x^{3/2}/3)$ as $x \to \infty$. $\mathrm{Ai}(x)$ has a countable number of real zeros, all negative.]

END OF PAPER