UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

## PAPER 70

## SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) State and prove the reciprocal theorem for two Stokes flows $\mathbf{u}_1$ and $\mathbf{u}_2$ with body forces $\mathbf{f}_1$ and $\mathbf{f}_2$ in a domain $V$ with boundary $\partial V$.

(b) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow.

Use this representation, explaining your choice of trial harmonic potentials, to determine the velocity field due to a rigid sphere of radius $a$ moving with velocity $\mathbf{U}+\boldsymbol{\Omega}\wedge\mathbf{x}$ through unbounded fluid of viscosity $\mu$ that is otherwise at rest. Determine also the stress field $\boldsymbol{\sigma}$ when $\mathbf{U}=\mathbf{0}$.

[*You may assume below that* $\boldsymbol{\sigma} = -\dfrac{3\mu}{2a^2}\left\{\mathbf{Ux}+\mathbf{xU}+(\mathbf{U}\cdot\mathbf{x})(\mathbf{I}-2\mathbf{xx}/a^2)\right\}$ *on* $r=a$ *when* $\boldsymbol{\Omega}=\mathbf{0}$ ]

(c) A spherical micro-organism of radius $a$ has the same density as the surrounding fluid and swims by using a surface layer of tiny flagella to prescribe a relative velocity $\mathbf{u}_s(\mathbf{x})$ between the fluid just outside the organism and the rigid body of the organism. Hence if the velocity of the organism is $\mathbf{V}+\boldsymbol{\omega}\wedge\mathbf{x}$, with $\mathbf{V}$ and $\boldsymbol{\omega}$ constants, then the fluid velocity immediately outside the organism is $\mathbf{V}+\boldsymbol{\omega}\wedge\mathbf{x}+\mathbf{u}_s(\mathbf{x})$.

Use the reciprocal theorem (with $\mathbf{f}=\mathbf{0}$) to determine $\mathbf{V}$ and $\boldsymbol{\omega}$ as integrals of $\mathbf{u}_s(\mathbf{x})$. Evaluate $\mathbf{V}$ and $\boldsymbol{\omega}$ for the case

$$\mathbf{u}_s(\mathbf{x}) = A(\mathbf{k}\wedge\mathbf{x})\wedge\mathbf{x}/a^2 + B(\mathbf{k}\cdot\mathbf{x})^2\mathbf{k}\wedge\mathbf{x}/a^3\ ,$$

where $A$ and $B$ are constants and $\mathbf{k}$ is a unit vector.

[*You may use* $\displaystyle\int_{r=a} x_i x_j x_k x_l \,\mathrm{d}S = \frac{4\pi a^6}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$ .]

**2** A rigid body is surrounded by unbounded fluid of viscosity $\mu$ that is otherwise at rest. If the body rotates with angular velocity $\mathbf{\Omega}$, explain why the force $\mathbf{F}$ it exerts on the fluid is given by $\mathbf{F} = \mu \mathbf{P} \cdot \mathbf{\Omega}$, where $\mathbf{P}$ is a tensor that depends only on the size and shape of the body.

Use the reciprocal theorem (which may be quoted without proof) to deduce the couple exerted by the body when it translates with velocity $\mathbf{U}$.

A slender wire of radius $\epsilon L$ has centreline $\mathbf{X}(s,t)$, where $\epsilon \ll 1$ and $s \in (-L, L)$ is the arc-length. Show briefly, under approximations to be specified, that when the wire moves through otherwise stationary unbounded fluid with velocity $\mathbf{V}(s) = \partial \mathbf{X}/\partial t$, it experiences a force $\mathbf{f}(s)$ per unit length, where

$$\mathbf{f}(s) = \frac{4\pi\mu}{\ln \epsilon^{-1}} \left(\mathbf{I} - \tfrac{1}{2}\mathbf{X}'\mathbf{X}'\right) \cdot \mathbf{V} \quad \text{and} \quad \mathbf{X}' = \partial \mathbf{X}/\partial s \ .$$

[*The Oseen tensor is* $\mathbf{J} = \dfrac{1}{8\pi\mu}\left(\dfrac{\mathbf{I}}{r} + \dfrac{\mathbf{xx}}{r^3}\right)$ .]

Suppose the wire is one turn of a rigid helix given in Cartesian coordinates by

$$\mathbf{X}(s) = a(\cos\theta, \sin\theta, \alpha\theta), \quad -\pi \leqslant \theta \leqslant \pi,$$

where $a$ and $\alpha$ are constants. Explain why, if $\alpha = 0$, the tensor $\mathbf{P}$ vanishes.

If instead $\alpha \ll 1$, show that, correct to $O(\alpha)$, slender-body theory gives

$$\mathbf{P} = -\frac{2\pi}{a \ln \epsilon^{-1}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\theta} \left(\mathbf{X} \wedge \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\theta}\right) \mathrm{d}\theta \ .$$

Deduce that, to the same level of accuracy, $\mathbf{P}$ is traceless.

Find $\mathbf{P}$ to $O(\alpha)$, showing that it is diagonal with respect to these axes.

[*You may use* $\displaystyle\int_{-\pi}^{\pi} \theta \sin\theta \cos\theta \, \mathrm{d}\theta = -\frac{\pi}{2}$ .]

**3** A vertical wall is coated with a thin liquid film of thickness $h(z)$, viscosity $\mu$ and density $\rho$, where $z$ is the vertical coordinate. Insoluble surfactant with concentration $C(z)$ resides on the surface of the film and reduces its surface tension according to the equation $\gamma(C) = \gamma_0 - AC$, where $\gamma_0$ and $A$ are positive constants. Assume that the variations of $h$ and $C$ are such that lubrication theory is valid.

Show that the vertical flux $j$ of surfactant is given by

$$j = -\left(\frac{AhC}{\mu} + D_s\right)\frac{\mathrm{d}C}{\mathrm{d}z} + \frac{h^2C}{2\mu}\left(-\rho g + \frac{\mathrm{d}}{\mathrm{d}z}\left(\gamma\frac{\mathrm{d}^2h}{\mathrm{d}z^2}\right)\right),$$

where $D_s$ is the surface diffusivity of the surfactant. Obtain the corresponding expression for the vertical flux $q$ of liquid.

(a) The surfactant evaporates from the film with a flux $E\,C(z)$ per unit height, but there is no evaporation of liquid. The consequent Marangoni stresses maintain a steady film, which extends from a large reservoir at $z = 0$, where $C = C_0 < \gamma_0/A$ and $C_0$ is a constant, up to a finite height $z = z_N$, where $h = 0$.

Write down equations describing conservation of liquid and surfactant in this steady state. Assuming that capillary pressure and diffusion can be neglected, show that

$$\frac{\mathrm{d}C}{\mathrm{d}z} = -\alpha h \quad \text{and} \quad \frac{\mathrm{d}(h^2C)}{\mathrm{d}z} = -\beta C\ ,$$

where $\alpha$ and $\beta$ are constants to be determined. By first solving these equations for $h(C)$, or otherwise, find $h(z)$ and $C(z)$, and show that

$$z_N = \frac{3}{4}\left(\frac{5A^2C_0^2}{\rho g\mu E}\right)^{1/3}.$$

Explain whether or not it is appropriate to neglect the capillary pressure near either $z = 0$ or $z = z_N$?

(b) Suppose now that the surfactant does not evaporate. Instead, a steady film is maintained by a constant flux of surfactant-free liquid with $h = h_\infty$ and $C = 0$ a long way above the reservoir. Assuming again that capillary pressure and diffusion can be neglected, show that the steady film thickness jumps suddenly by a factor of $4^{1/3}$ at $z = 2^{1/3}AC_0/(\rho g h_\infty)$. Interpret this result physically.

What physical effects might smooth this jump?

**4** Fluid of viscosity $\mu$ occupies the region $z > 0$ above a horizontal rigid boundary at $z = 0$. The position of a thin rigid plate of length $2L$ (and infinite extent in the $y$-direction) is given by $z = h_0(t) + \theta(t)[x - x_0(t)]$ with $x_0 - L \leqslant x \leqslant x_0 + L$, where $0 < h_0 \ll L$ and $0 \leqslant \epsilon \equiv \theta L/h_0 < 1$.

Interpret the conditions on $\epsilon$ geometrically. Express the gap $h$ between the plate and the boundary in terms of $h_0$, $\epsilon$ and the dimensionless coordinate $s = (x - x_0)/L$.

The force $(F_x, 0, F_z)$ and couple $G$ (each per unit length in the $y$-direction) exerted by the plate on the fluid are given by

$$\begin{pmatrix} F_x \\ F_z \\ G/L \end{pmatrix} = \mu \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} U \\ W \\ \Omega L \end{pmatrix} ,$$

where $U = \mathrm{d}x_0/\mathrm{d}t$, $W = \mathrm{d}h_0/\mathrm{d}t$ and $\Omega = \mathrm{d}\theta/\mathrm{d}t$, and the couple $G$ is calculated about the centre of the plate.

Use symmetry to show that $a_{12} = a_{32} = 0$ when $\epsilon = 0$. Use lubrication theory to calculate the rest of the dimensionless matrix $\{a_{ij}\}$ for $\epsilon = 0$.

Now consider the case $W = \Omega = 0$ and $\epsilon \ll 1$. Show that the pressure under the plate is given by

$$p = \frac{6\mu UL}{h_0^2}\left\{\tfrac{1}{2}\epsilon(1 - s^2) + \epsilon^2(s^3 - s) + O(\epsilon^3)\right\} .$$

[*It may be helpful to work in the frame where the plate is at rest and to use the binomial expansion* $(1 + u)^{-n} = 1 - nu + \frac{1}{2}n(n+1)u^2 + O(u^3)$ *to simplify integrations.*]

Deduce the leading-order values of $a_{21}$ and $a_{31}$. Why is $G \ll F_z L$? Explain briefly the relationships between the force exerted tangentially by the plate, the force exerted tangentially on the plane and the horizontal force $F_x$.

Given that $a_{32} = -\frac{8}{5}\epsilon(L/h_0)^3$, write down the complete matrix $\{a_{ij}\}$ with each element at leading order in $\epsilon$.

The plate has mass per unit area $m/(2L)$ and settles under gravity from a position with $h_0 \ll L$ and $\epsilon \ll 1$ at $t = 0$. By considering the force and moment balances at leading order, show that

(i) $h_0 \propto t^{-1/2}$ as $t \to \infty$,

(ii) $\theta(t) = \theta(0)[h_0(t)/h_0(0)]^3$,

(iii) the plate drifts a horizontal distance $\Delta x_0 = -L^2\theta(0)/h_0(0)$ as $h_0 \to 0$.

## END OF PAPER