UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 1:30 pm to 4:30 pm

PAPER 7

TOPICS IN ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Show that any continuous function of two variables $f : [0,1]^2 \to \mathbb{R}$ can be written in terms of continuous functions of one variable and addition.

2

Show that the following two statements are equivalent.

(A) If (E, d) is a complete metric space, then E cannot be written as the union of a countable collection of closed sets with empty interior.

(B) Suppose that X and R are sets with $R \subseteq X \times X$ and such that

$$\{y \in X \,:\, (x, y) \in R\} \neq \emptyset$$

for all $x \in X$. Then there exists a function $G : \mathbb{Z}^{++} \to X$ such that $\big(G(n), G(n+1)\big) \in R$ for all $n \geqslant 1$.

3

Define the terms *convex set*, *closed convex hull* and *extreme point* as used in the statement that every compact convex set in a normed space over $\mathbb{R}$ is the convex hull of its extreme points.

Prove the statement.

State and prove Carathéodory's lemma about convex sums in $\mathbb{R}^n$.

Consider the space $l^1(\mathbb{R})$ of real sequences $\mathbf{x}$ with $\sum_{j=1}^{\infty} |x_j|$ convergent and norm $\|\mathbf{x}\|_1 = \sum_{j=1}^{\infty} |x_j|$. Find the extreme points of the closed unit ball Σ and show that not every point of Σ is a finite convex combination of extreme points.

Show that if K is a compact convex set in $\mathbb{R}^n$ whose extreme points form a closed set, then every point in K is in the convex hull of $n+1$ of its extreme points. Show by means of an example that we cannot replace $n+1$ by n.

If K is a compact convex set in $\mathbb{R}^n$, is it true that K is the convex hull of finitely many extreme points? Give a proof or counterexample.

4

Define the space of *distributions* $\mathcal{D}'(\mathbb{T})$ and the notion of *convergence in distribution*. Define the *derivative* T' of a distribution T showing that is is indeed a distribution. Characterise the members of $\mathcal{D}'(\mathbb{T})$ in terms of their Fourier series.

State and prove a necessary and sufficient condition for a distribution T to be the derivative of another distribution.

Define the *support* of a distribution showing that it is well defined object. (You may use any theorems you wish on partitions of unity provided they are clearly stated.) Show that if T is a distribution $\operatorname{supp} T' \subseteq \operatorname{supp} T$ and give an example where $\operatorname{supp} T' \neq \operatorname{supp} T$.

Characterise the distributions whose support is a single point. (If you use any form of Taylor's theorem you should prove it.)

Show that given any $n \geqslant 0$ and and $\epsilon > 0$ we can find a $g \in \mathcal{D}$ such that $\operatorname{supp} g \subseteq [-\epsilon, \epsilon]$, $g^{(n)}(0) = 1$ and $\|g^{(r)}(t)| \leqslant \epsilon$ for all t and all $0 \leqslant r \leqslant n$.

Let δ_x have its usual meaning and let $a_j \in \mathbb{C}$ $[j \geqslant 1]$.

(i) Show that $\sum_{j=1}^{m} a_j \delta_{1/j}$ converges in distribution as $m \to \infty$ if $\sum a_j$ is absolutely convergent.

(ii) If E is a closed subset of $\mathbb{T}$, is it true that there always exists a distribution T with $\operatorname{supp} T = E$? Give a proof or counter example.

END OF PAPER